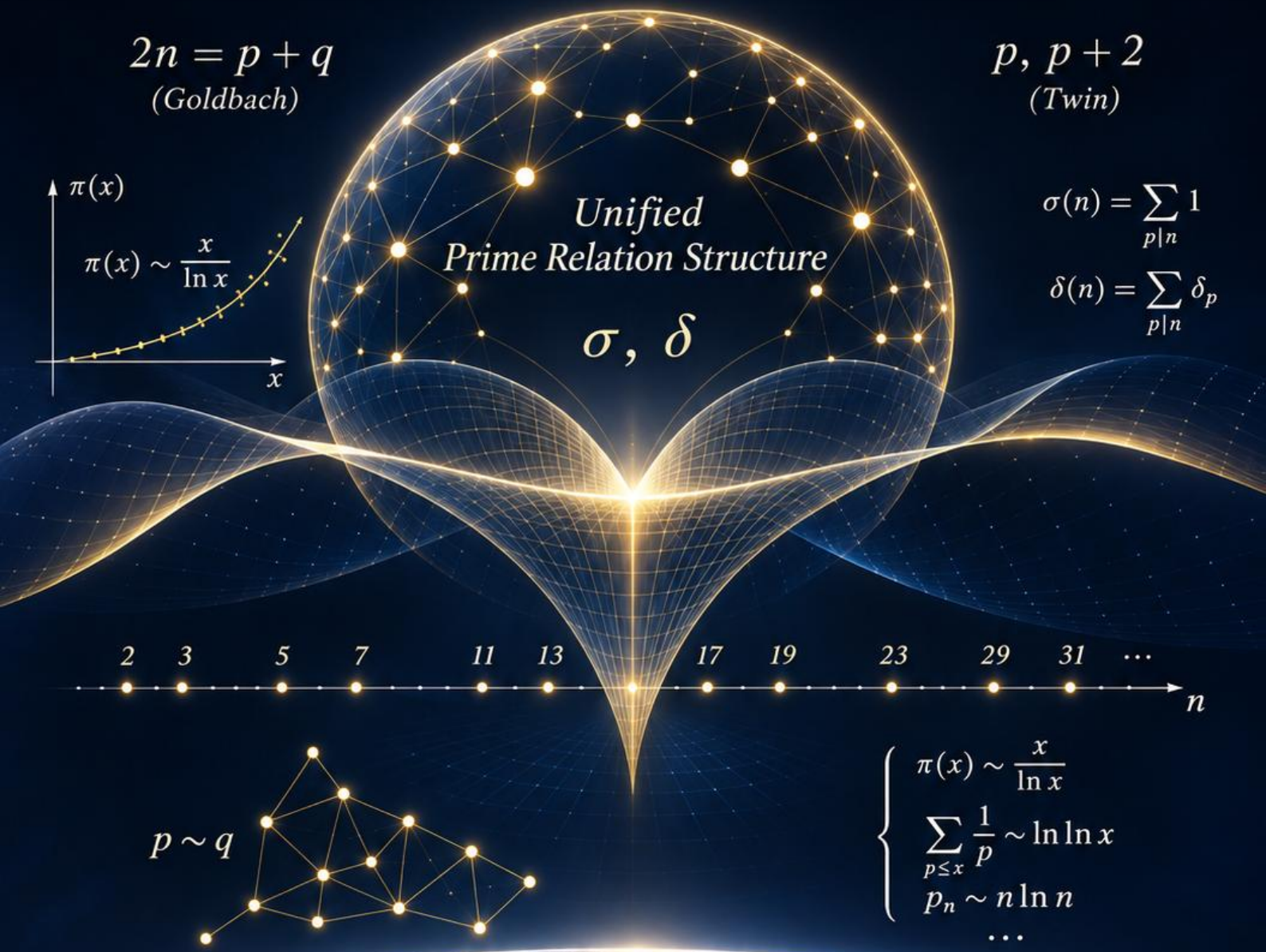
Unified Prime Relations Theory

— Proofs of the Goldbach and Twin Prime Conjectures and the Common Matrix of Classical Prime Propositions



Yifei Ren

尘缘聚散原无定

莫向空门觅旧根

Worldly ties gather and part, with no fixed course;

Seek not old roots within the gate of emptiness.

Unified Prime Relations Theory

Unified Prime Relations Theory · English Edition

Author Yifei Ren（任毅飞）

Author status Independent Researcher

Publication date November 12, 2026

Email zentiger@qq.com

ORCID 0009-0009-1373-9609

WeChat ryfweixin

GitHub (interactive teaching website and software for this book) https://github.com/qiaoranjushi/ghtmath

Highly Recommended Interactive Classroom https://qiaoranjushi.github.io/ghtmath/

Book DOI https://doi.org/10.5281/zenodo.24144189

This book is currently a preprint and is not under contract with a publisher. If you believe it may be suitable for publication, I would be glad to hear from you and discuss possible collaboration.

I am an independent researcher. If you would like to support my research financially, I would be sincerely grateful for your support.

Intellectual background The foundational ideas behind this book grew out of *Foundations of Mathematics: An Introduction to Generative History Theory (GHT)*（《数学的地基：GHT 生成履历理论概论》）

https://doi.org/10.5281/zenodo.23662772

Copyright and license © 2026 Yifei Ren. Unless otherwise indicated, this book is licensed under the Creative Commons Attribution–NonCommercial 4.0 International License (CC BY-NC 4.0). Subject to the attribution and noncommercial-use conditions of that license, anyone may copy, distribute, display, adapt, and build upon the contents of this book. Third-party material and material carrying separate copyright notices remain subject to their respective terms.

arXiv endorsement request I am currently seeking endorsement to submit to the arXiv math.NT (Number Theory) category. If you are eligible to endorse submissions in this category and are willing to support my submission, please visit:

https://arxiv.org/auth/endorse?x=X4R44M

Thank you for your time and help.

Author's Note

I am still a beginner in mathematics and only at the beginning of a much longer exploration. This book records my present understanding, structural attempts, and current results concerning prime relations, their unification, and the associated proof problems; it is not intended as a final word immune to revision. Because my mathematical knowledge, research experience, and exposition are still developing, some definitions may be sharpened further, some arguments may benefit from stronger presentation or formalization, and there may be errors that I have not yet recognized. One of the essential strengths of mathematics is precisely that its claims can be made public, checked independently, criticized, corrected, and rebuilt. I therefore warmly welcome comments, objections, counterexamples, corrections, suggestions for clearer formulations, and better proofs from readers and researchers. Careful and specific criticism will be treated not as an obstacle to this work, but as an important part of its continued refinement and development.

Contents

Author's Note iii
Preface: From Isolated Prime Mysteries to a Unified World of Relations 1
I. Prime Numbers: Among the Oldest Objects in Mathematics, Yet Among Its Youngest Problems 1
II. From Euclid and Euler to Riemann: Prime Numbers Move from "Atoms" to Global Structure 2
III. Goldbach: From Prime Identity to Fixed-Sum Relations 3
IV. Twin Primes and Polignac: The Other Fundamental Axis Is Fixed Difference 3
V. Landau's Four Problems: A Concentrated Naming of Prime-Number Difficulties at the Beginning of the Twentieth Century 5
VI. The Twentieth and Twenty-First Centuries: Great Advances Keep Compressing the Problems Toward Their Lowest-Order Core 6
VII. Two Isolated Peaks Enter One Mountain Range: The Core Results of This Book 7
VIII. The Real Turning Point: From "One Conjecture, One Language" Back to a Common Matrix 9
IX. Part I: The Fixed Arithmetic Identity Spectrum and the Atomic Relation Matrix 10
X. Part II: Finite Moving Horizons—the Finite Entry Point to Infinite Relations 11
XI. Part III: Relation Slices, Derived Geometry, and Quantifier Structures 11
XII. Part IV: Finite-Control Geometry and the Terminal-Saturation Structure of Goldbach 12
XIII. Part V: The Exact Architecture of Infinite Relations and the Persistent Non-Extinction Structure of Twin 12
XIV. Part VI: Unified Compilation of Classical Propositions and the Boundaries of Methods 13
XV. Part VII: RenSpectrum—Bringing the Theory into an Executable World 13
XVI. From Two Classical Propositions to a Theory of Relations 14
XVII. Before Reading 15
Terminology Glossary — Unified Prime Relations Theory 16
Part I — The Fixed Arithmetic Identity Spectrum and the Atomic Relation Matrix 18
Introduction — From "Many Prime Problems" Back to "One Fixed Spectrum" 19
Abstract 20
Chapter 1 — The First Object: The Fixed Arithmetic Identity Spectrum 23
1.1 The Integer Line Comes Before Any Particular Relation 23
1.2 The Primality Indicator Is Only a Projection of the Identity Spectrum 24
1.3 A Single Spectrum Can Carry Multiple Fields 25
Chapter 2 — Identity and Relation: Two Levels That Cannot Be Interchanged 26
2.1 Relations Select Positions; They Do Not Produce Positional Identity 26
2.2 Three Levels 26
Chapter 3 — Why Atomic Coordinates Are Still Needed 29
3.1 The Integer Spectrum and the Atomic Spectrum Are Two Views of the Same Object 29
Chapter 4 — Two Basic Operations on the Shared Coordinates 31

Chapter 5 — Multiplicative Compression: Exact Transport of Odd Multiplication 32
Chapter 6 — Additive Expansion: Expressing Two-Odd-Integer Relations in the Same Coordinates 33
Chapter 7 — Irreducible Atoms: Odd Primes Emerge from Multiplication Itself 34
7.1 The Position of the Prime 2 35
Chapter 8 — Unique Atomic Factorization and Multiplicative History 36
8.1 Correspondence Between Atomic Factorization and Identity Fields 36
Chapter 9 — Atomic Rays: Linearization of Multiplicative Generation 38
Chapter 10 — Coverage of Non-Atoms and Recursive Generation of the Next Atom 39
10.1 Coverage Is Not a Partition 39
Chapter 11 — From Atomic Points to Relational Objects 42
Chapter 12 — The Law of Relation Birth: One New Atom Produces an Entire Row 43
Chapter 13 — Ray–Relation Duality: Two Expansions Triggered by the Same Atom 45
Chapter 14 — Combinatorial Amplification and Block Seams 47
14.1 A Relation Seam Is Not a Single Point 47
Chapter 15 — Persistent Embedding: Why Historical Relations Are Never Lost 49
15.1 Confirmed Relations and Candidate Relations Must Be Distinguished 49
Chapter 16 — Translation of Relation Rows: History Is Not Random Resampling 51
Chapter 17 — Lossless Sum–Difference Coordinates 52
17.1 Realizable Lattice Points 53
17.2 Fixed Sum and Fixed Difference Are Not Two Separate Worlds 53
Chapter 18 — From “Relation Instances” to “Relation Slices” 55
Chapter 19 — From Two-Atom Slices to General Position Maps 56
19.1 A Fixed Sum as a Position Map 56
19.2 A Fixed Difference as a Position Map 57
19.3 A Finite Pattern as a Bundle of Position Maps 57
19.4 An Arithmetic-Progression Bundle as a Parameterized Pattern 57
19.5 Affine Relations 57
19.6 Polynomial Pullback 57
Chapter 20 — Windows, Consecutive Primes, and Complete Factor Identity Are Also Slices 59
20.1 Window Occupancy 59
20.2 Empty-Window Readout 59
20.3 Indexing Consecutive Primes 59
20.4 Complete Factor-Set Slices 60
Chapter 21 — Unified Relation Normal Form 61
21.1 A Proposition Adds Quantifiers Only at the Outermost Layer 61
Chapter 22 — Relation Fidelity: Why One Cannot Compress Too Early to a Single Number 63
22.1 Minimal Information in a Relation State 63
Chapter 23 — The Natural Entry Point to Higher-Order Relations 64
Chapter 24 — Formal Definition of the Finite-Layer Matrix 65
Chapter 25 — Zeroth-Order Closure of Matrix Dynamics 66
Chapter 26 — A Complete Small Example 67
26.1 Fixed Integer Identity Spectrum 67

26.2 Atomic Rays 67
26.3 Relation-Birth Layers 68
26.4 Why a Sum Collision Cannot Be Used to Compress Away the Relation 68
Chapter 27 — Summary Table of Foundational Theorems in Part I 70
Foundational Theorem A — Fixed Identity 70
Foundational Theorem B — Multiplicative Compression 70
Foundational Theorem C — Atomic Equivalence 70
Foundational Theorem D — Recursive Generation of Atoms 70
Foundational Theorem E — Full-Row Birth of Relations 70
Foundational Theorem F — Persistent Inheritance 70
Foundational Theorem G — Translation of Relation Rows 70
Foundational Theorem H — Lossless Sum–Difference Coordinates 70
Foundational Theorem I — Relation-Slicing Principle 70
Chapter 28 — From Relation Slices to Quantifier Structures 72
Chapter 29 — How to Read the Relation Matrix 73
Conclusion — From a List of Primes to a Readable Relation Spectrum 73
Part II — Finite Moving Horizons, Relation Reconstruction, and Permanent Preservation 76
Abstract 77
Chapter 1 — Why Fixed Identity Cannot Replace Relation 80
Chapter 2 — Relation Reconstruction: What It Means to Truly Preserve a Relation 82
2.1 Sum–Difference Coordinates Are Lossless 83
2.2 The Sum Alone Is Not Lossless 83
2.3 This Is Exactly Where the Value of Relation Theory Lies 83
Chapter 3 — Persistent Relation Filtering: Why Old Relations Are Never Lost 85
Chapter 4 — Candidate Relations and Confirmed Relations 88
Chapter 5 — Relation-Faithful States: Why One Count Is Usually Not Enough 90
5.1 An Explicit Example 90
Chapter 6 — Unified Moving Horizons: How a Finite Window Receives Zero-Error Atomic Certification 92
Chapter 7 — Prime Squares: The Natural Discrete Boundaries of Moving Horizons 95
Chapter 8 — Genuine Death Layers: A Composite Dies for the First Time Only Once 97
Chapter 9 — Relation-Faithful Sieves: From Endpoint Death to Relation Death 99
9.1 Fixed-Sum Relation Slots 99
9.2 Terminal Exactness 100
Chapter 10 — Terminal Certification of Finite Relations 101
Chapter 11 — Promotion from Candidate Shells to the Permanent Relation Skeleton 103
Chapter 12 — Relation Histories: An Object Has Not Only a Result but a Generative History 104
Chapter 13 — Birth and Death Are Strictly Misaligned in Scale 106
Chapter 14 — A Moving Horizon Does Not Create Relations; It Reads Them Terminally 107
Chapter 15 — Unified Horizons and Readout-Adapted Horizons 108
15.1 Adapted Terminal Horizon for a Fixed-Sum Readout 108
15.2 Adapted Terminal Horizon for Fixed Finite Endpoints 108
15.3 Adapted Terminal Horizon for a Window Readout 108

15.4 Adapted Terminal Horizon for a General Finite Relation 109
Chapter 16 — Unified Normal Form for Relation Reconstruction 110
Chapter 17 — How Finite Horizons Enter Infinite Quantifiers 111
17.1 Infinite Recurrence 111
17.2 Parameterwise Universal Fibers 111
17.3 Window Universality 112
17.4 Arbitrary-Length Relations 112
17.5 Infinitely Many Parameters 112
17.6 Indexwise Universal Predicates 112
Chapter 18 — Finite-to-Infinite Entry Theorem 114
Chapter 19 — How Different Global Quantifiers Enter the Same Matrix 115
Chapter 20 — The Finite Foundation of Batch Proof 116
Chapter 21 — Four Basic Forms of Finite Closure 117
21.1 Finite-Quotient Closure 117
21.2 Propagation Closure 117
21.3 Cumulative-Potential Closure 117
21.4 Bad-State Exclusion Closure 117
Chapter 22 — Four Basic Examples 118
22.1 Fixed Sum 118
22.2 Fixed Gap 118
22.3 Square Window 118
22.4 Finite Pattern 118
Chapter 23 — Unified Dynamics: Birth, Elimination, Promotion, Persistence, and Readout 120
Chapter 24 — Summary Table of Foundational Theorems in This Part 122
Foundational Theorem F — Persistent Inheritance of Lower-Order Relations 122
Foundational Theorem G — Atomic Certification by a Finite Moving Horizon 122
Foundational Theorem H — Fixed History in a Square Horizon 122
Foundational Theorem I — Unique Genuine Death 122
Foundational Theorem J — Terminality of Finite Relations 122
Foundational Theorem K — Irreversibility of Promotion 122
Foundational Theorem L — Finite-to-Infinite Entry 122
Foundational Theorem M — Counting Criterion for Infinite Recurrence 123
Foundational Theorem N — Fiber Criterion for Parameterwise Universality 123
Chapter 25 — The Relation Layer Cannot Be Erased by the Fixed Spectrum 124
Chapter 26 — Discipline Between Finite Layers and Global Propositions 125
Chapter 27 — General Technical Interface for Relation-Faithful Sieves 126
Chapter 28 — From Finite Terminality to Unified Prime Relations Theory 127
Conclusion — The Mathematical Entry Point to an Infinite Relation Is Finite Terminality 127
Part III Relation Slices, Derived Geometry, Quantifier Structures, and Batch Proof 129
Abstract 130
Part I The Binary Atomic Relation Plane and Derived Geometry 134
Chapter 1 The Binary Atomic Relation Plane: A Common Matrix for Sum and Difference 135
Chapter 2 Three Natural Directions in the Relation Plane 137

2.1 Fixed Sum: Vertical Slices 137
2.2 Fixed Difference: Horizontal Slices 137
2.3 Fixed Larger Atom: An Oblique Relation Star 138
2.4 Fixed Product: Hyperbolic Leaves of the Same Plane 139
2.5 Total Multiplicative-Depth Identity 140
Chapter 3 Shell Intersections, Orthogonal Generation, and Reflection–Translation 143
3.1 Two Families of Endpoint Shells 143
3.2 Factorization of Shell Depth 144
3.3 Shell Geometry of the Birth of New Atoms 145
3.4 Orthogonal Generating Directions 145
3.5 Prime Gaps as a Local Metric of the Relation Grid 146
3.6 Reflection and Translation: Two Self-Intersections of the Same Atomic Set 146
Chapter 4 Lower-Order Witness Discipline: Unification Cannot Be Purchased by Losing the Relation 149
Part II Lower-Order Witness Fields and Finite Atomic Constellations 149
Chapter 5 The Finite Goldbach Witness Field 150
Chapter 6 Twin as Persistence Across Adjacent Fixed-Sum Witness Columns 151
Chapter 7 Exact Identity Between Fixed-Difference Counts and Witness Persistence 153
Chapter 8 Multi-Column Persistence and General Prime Constellations 155
Chapter 9 Temporal and Spatial Directions 157
Part III General Relational Language and Relation Normal Form 157
Chapter 10 Definition of a General Relation Slice 158
Chapter 11 Unified Relation Normal Form 160
Chapter 12 Fixed Sum, Fixed Difference, and Consecutive Fixed Difference 161
12.1 Fixed Sum 161
12.2 Fixed Difference 161
12.3 Consecutive Fixed Difference 161
Chapter 13 Weighted Reflections and Affine Relations 162
13.1 Lemoine / Lévy-Type Weighted Reflection 162
13.2 General Affine Relations 162
Chapter 14 Finite Patterns and Prime Tuples 163
Part IV Higher-Order Affine Organization and Arithmetic Atomic Chains 163
Chapter 15 Green–Tao Arithmetic Atomic Chains 164
Chapter 16 An Arithmetic Chain as a Concatenation of Equal-Gap Binary Atomic Relations 166
Chapter 17 Three-Term Chains and Fixed-Sum Reflection 167
Chapter 18 Fixed-Sum Mirror Folding of an Arithmetic Chain of Arbitrary Length 168
Chapter 19 Square Horizons for Arithmetic Chains 169
19.1 A Long Chain May Cross a Square Boundary 170
Chapter 20 Single-Prime Forbidden Classes for Arithmetic Chains 171
Chapter 21 Chain Capacity at a Finite CRT Layer 173
Chapter 22 Intrinsic Step Capacity of Long Chains 174
Chapter 23 Finite-Layer Nonemptiness Model 176
Chapter 24 Birth Law for Higher-Order Organizations 177

Chapter 25 The Green–Tao Theorem in the Unified Relational Language 179
25.1 Reference Theorem 179
Chapter 26 Rigorous Transfer from Arithmetic Chains to High Goldbach Multiplicity 180
Chapter 27 A Five-Term Arithmetic-Progression Example 182
Part V Windows, Indices, Pullbacks, and Complete Factor Identities 182
Chapter 28 Window Relations 183
28.1 Legendre Windows 183
28.2 Oppermann Double Windows 183
28.3 Brocard Windows Between Consecutive Prime Squares 183
Chapter 29 Relations Indexed by Consecutive Primes 185
29.1 Andrica-Type Relation 185
29.2 Firoozbakht-Type Relation 185
Chapter 30 Exponential and Polynomial Pullbacks 186
30.1 Mersenne-Type Exponential Pullback 186
30.2 Bunyakovsky-Type Polynomial Pullback 186
30.3 Schinzel H Polynomial Tuples 186
Chapter 31 Relations of Complete Factor Identity 187
31.1 Grimm-Type Factor Matching 187
31.2 Erdős–Woods-Type Identity Blocks 187
Part VI Quantifier Structures: Universality, Infinitude, and Other Global Requirements 188
Chapter 32 Quantifiers Are Not Decoration: They Determine the Global Type of a Proposition 189
32.1 Finite Readout 189
32.2 Infinite Recurrence 189
32.3 Parameterwise Universal Fibers 189
32.4 Window Universality 189
32.5 Arbitrary Length 190
32.6 Parameter Infinitude 190
32.7 Indexwise Universal Predicates 190
32.8 Identity-Block Separation 191
Chapter 33 Why “Universal” and “Infinite” Must Be Kept Separate 192
33.1 Universal Fibers Require “Every Column Nonempty” 192
33.2 Infinite Recurrence Requires “The Total Number of Hits Keeps Growing” 192
Part VII Batch Proof and Global Closure 192
Chapter 34 Batch Proof: One Proof Covers an Entire Parameter Family 193
34.1 Batch Proof Is Not Batch Enumeration 193
Chapter 35 Batch Decision for Universal Fibers 195
35.1 Uniform Positive Lower Bound 195
35.2 Exclusion of Failure States 195
35.3 Exclusion of a Least Counterexample 195
Chapter 36 Batch Decision for Infinite Recurrence 196
36.1 Direct Growth 196
36.2 Cumulative Net Potential 196
Chapter 37 Four Basic Modes of Closure 198

37.1 Finite-Quotient Closure 198
37.2 Propagation Closure 198
37.3 Cumulative-Potential Closure 198
37.4 Bad-State Exclusion Closure 198
37.5 Local-Finite / Global-Assembly Closure 198
Part VIII Relation Equivalence, Executable Syntax, and Master Theorems 200
Chapter 38 Relation Equivalence and Relation Difference 201
38.1 Goldbach in Atomic Coordinates and Ordinary Prime Coordinates 201
38.2 Fixed Difference and Horizontal Slices of the Relation Plane 201
38.3 Equal Counts Do Not Imply Relation Equivalence 202
Chapter 39 Minimal Operations of the Relational Language 203
R1 Position Mapping 203
R2 Identity Readout 203
R3 Finite Logic and Finite Comparisons 203
R4 Window Aggregation 203
R5 Outer Quantifiers 203
Chapter 40 From Mathematical Syntax to an Executable Relational Language 204
Chapter 41 Three Custom Relation Examples 205
41.1 Cross-Mirror Relation 205
41.2 Quadratic Double-Atomic Relation 205
41.3 Reflection–Translation Coupled Relation 205
Chapter 42 Relation Type and Quantifier Type as Two Independent Coordinates 207
Chapter 43 Unified Table of Classical Relations 208
Chapter 44 Minimal Completeness Conditions for Batch Proof 209
44.1 Relation Fidelity 209
44.2 Finite Local Correctness 209
44.3 Parameter Coverage 209
44.4 Quantifier Closure 209
Chapter 45 When Higher-Order Structure Is Genuinely Useful 211
Chapter 46 Table of the Fundamental Theorems of This Part 212
Fundamental Theorem O Losslessness of the Binary Atomic Relation Plane 212
Fundamental Theorem O1 Fixed-Product Hyperbolic Leaves 212
Fundamental Theorem O2 Total Multiplicative-Depth Identity 212
Fundamental Theorem O3 Shell-Intersection Bijection 212
Fundamental Theorem O4 Equivalence of Shell Growth and Relation Growth 212
Fundamental Theorem O5 Orthogonal Generation 212
Fundamental Theorem P Witness-Persistence Identity 213
Fundamental Theorem Q Multi-Column Persistence Identity 213
Fundamental Theorem R Equal-Gap Chain Concatenation 213
Fundamental Theorem S Mirror Folding of Arithmetic Chains 213
Fundamental Theorem T Number of Forbidden Classes at One Prime Layer 213
Fundamental Theorem U Finite CRT Chain Capacity 213
Fundamental Theorem V Necessary Step Condition for Long Chains 213

Fundamental Theorem W Birth Law for Higher-Order Organizations 214
Fundamental Theorem X Infinite-Recurrence Criterion 214
Fundamental Theorem Y Universal-Fiber Criterion 214
Fundamental Theorem Z Batch-Certificate Principle 214
Fundamental Theorem Z1 Local-Finite / Global-Assembly Theorem 214
Conclusion One Spectrum Does Not Mean One Relation 215
Part IV Finite-Control Geometry, Local Obstruction, and Relation Sieves 219
Abstract 220
Part I Finite-Control Geometry 226
Chapter 1 From Relational Language to Control Geometry 227
Chapter 2 Relation Spaces of Arbitrary Finite Order 228
2.1 Prime-Gap Cells 228
Chapter 3 The Multiplicative-Depth Tensor 229
Chapter 4 Vertices, Edges, Faces, and Shallow Multiplicative Depth 230
Chapter 5 Relation Orbits 231
Chapter 6 Atomic-Vertex Preimage Theorem 232
Chapter 7 Finite Control Depth 233
Chapter 8 Phase Periodicity and the Finite Control Period 234
Part II Local Obstruction and Admissibility 235
Chapter 9 Unified Definition of Local Obstruction 236
Chapter 10 Local Admissibility Does Not Equal Global Occupancy 237
Chapter 11 Local Fingerprint of a Finite Offset Pattern 238
Chapter 12 Local Admissibility of a Pattern 240
Chapter 13 A Completely Obstructed Pattern 241
Chapter 14 Local Capacity of Arithmetic Patterns 242
Chapter 15 Local Capacity After Finitely Many Rounds 244
Part III The CRT Survival Tree and First Local Elimination 245
Chapter 16 From Finite Capacity to the CRT Survival Tree 246
16.1 Layerwise Control Moduli 246
16.2 Exact Branching of a Surviving Parent Node 246
16.3 Survival Proportion 247
16.4 Complete Obstruction Means Termination of the Entire Tree 248
Chapter 17 Conservation of First Local Elimination 250
17.1 First-Local-Elimination Sets 250
17.2 Exact Count of First Eliminations 250
17.3 From a Phase First Killer to an Integer First Killer 251
Part IV Genuine Death, Relation Alphabets, and Relation-Faithful Sieves 253
Chapter 18 Ordinary Divisibility and Genuine Death 254
Chapter 19 The Relation First Killer 255
Chapter 20 Relation Alphabets 256
20.1 Endpoint-Identity Letters 256
20.2 Relation Identity Words 256
20.3 Coarse Alphabet for Fixed Sums 256

20.4 Purified Alphabet for Twin in a Single Square Horizon 257
20.5 Relation Alphabets and Relation Fidelity 257
Chapter 21 The Relation-Faithful Sieve 259
Chapter 22 Why "How Many Remain?" Is Not a Sufficient State 260
Chapter 23 Fixed Sum as a Two-Point Reflection Correlation 262
Chapter 24 Finite-Modulus Data Do Not Determine Reflection Pairing 263
Chapter 25 Division of Labor Between Relation Sieves and Element Sieves 264
Part V Parity Channels of Multiplicative Depth 265
Chapter 26 Liouville Parity Identity 266
Chapter 27 Parity Flip Under Genuine Death 267
Chapter 28 Odd–Even Dual-Channel Transfer 268
Chapter 29 Four Relation Channels 269
Chapter 30 Upper Bound on Multiplicative Depth 270
Chapter 31 Cube-Root Certification of the Odd Channel 271
Chapter 32 The Hierarchical Difference Between Cube-Root Certification and Fixed-Sum Existence 272
Chapter 33 The $P1/P3$ Decomposition at the $X1/4$-Layer 273
Chapter 34 Why a Simple Monotone Proportion Is Not Enough 274
Chapter 35 From the Parity Obstacle to Reflection Localization 275
Part VI Fixed Sums: A Terminal-Saturation Relation Sieve 276
Chapter 36 From the General Relation Sieve to Fixed Sums 277
Chapter 37 Matrix-to-Subrelation Readout Inheritance 278
Chapter 38 Why Fixed Sum Is One of the Lowest-Order Exact Relations 279
Chapter 39 Finite History and Fixed-Sum Support 280
Chapter 40 Translation of Relation Rows 281
Chapter 41 Recurrence for Fixed-Sum Multiplicity 282
Chapter 42 Terminal-Horizon Operator 283
Chapter 43 Natural Terminal Freezing for Fixed Sums 284
Chapter 44 Natural Terminal Segments 285
Chapter 45 Returning to the Even Axis: Prime Gaps Determine Terminal-Segment Width 287
Chapter 46 Defects and the First Defect 288
Chapter 47 Reciprocal Motion Within a Fixed Sum 289
Chapter 48 Finite Relation Certificates 290
Part VII Exact Additive–Multiplicative Completion Objects 291
Chapter 49 Position Level: Complementary Identities of Atoms and Multiplicative Occupancy 292
Chapter 50 Temporarily Unread States Within a Finite Horizon 293
Chapter 51 Fixed Sum as Re-Pairing Only 294
Chapter 52 Relation-Slot Exhaustion Theorem 295
Chapter 53 Exact Slotwise Completion Identity 296
Chapter 54 Conservation of the Entire Relation-Slot Set 297
Chapter 55 Responsibility Decomposition of Double Multiplicative Occupancy 299
Chapter 56 Temporal Offset Between Atomic Birth and Exclusive Multiplicative Action 300
Chapter 57 Concrete Relation Slots: $N = 12$ 301

Chapter 58 Concrete Relation Slots: $N = 16, N = 18, N = 20, N = 40$ 302
58.1 $N = 16$ 302
58.2 $N = 18$ 302
58.3 $N = 20$ 303
58.4 $N = 40$ 303
Part VIII Terminal Saturation, Universal Readout, and the Standard Toolset 304
Chapter 59 Goldbach Relation Completeness and Its Exact Position 305
Chapter 60 The Threefold Terminal Equivalence 307
Chapter 61 Readout–Terminal Placement Theorem 308
Chapter 62 Complete Finite Sections 309
Chapter 63 Support and Multiplicity 310
Chapter 64 Eight Standard Relation-Sieve Operators 311
64.1 Birth Operator 311
64.2 Projection Operator 311
64.3 Terminal Operator 311
64.4 Defect Operator 311
64.5 First-Defect Operator 311
64.6 Opportunity Operator 311
64.7 Certificate Operator 311
64.8 Saturation Operator 312
Chapter 65 Terminal-Saturation Type and Persistent Non-Extinction Type 313
65.1 Terminal-Saturation Type 313
65.2 Persistent Non-Extinction Type 313
Chapter 66 Why Fixed Sum and Fixed Difference Cannot Use the Same Terminal Logic 314
Part IX Capability Boundaries and General Interfaces of Relation Sieves 315
Chapter 67 Division of Labor Between Finite Terminality and Universal Closure 316
Chapter 68 Simple Cardinality Capacity Cannot Replace Reflection Localization 317
Chapter 69 Four Reusable Interfaces of the Relation-Faithful Sieve 318
69.1 Relation-Fidelity Interface 318
69.2 Atomic-Birth / Relation-Row Interface 318
69.3 Multiplicative-Depth Channel Interface 318
69.4 Failure-Certificate Interface 318
Chapter 70 How a New Relation Enters Finite-Control Geometry 319
Chapter 71 Table of Fundamental Theorems in This Part 320
Fundamental Theorem AA Atomic-Vertex Preimage 320
Fundamental Theorem AB Local Admissibility of Finite Patterns 320
Fundamental Theorem AC CRT Survival Tree 320
Fundamental Theorem AD Conservation of First Local Elimination 320
Fundamental Theorem AE Uniqueness of Genuine Death 320
Fundamental Theorem AF Relation First Killer 320
Fundamental Theorem AG Relation Alphabet 321
Fundamental Theorem AH Cube-Root Certification of the Odd Channel 321
Fundamental Theorem AI Odd-Channel Decomposition at $X1/4$ 321

Fundamental Theorem AJ Terminal Freezing of Fixed Sums 321
Fundamental Theorem AK Partition by Natural Terminal Segments 321
Fundamental Theorem AL Moving-Intersection Equivalence 321
Fundamental Theorem AM Relation-Slot Exhaustion 321
Fundamental Theorem AN Exact Slotwise Completion 321
Fundamental Theorem AO Conservation of the Entire Relation-Slot Set 321
Fundamental Theorem AP Decomposition by Double-Multiplicative Responsibility 321
Fundamental Theorem AQ Equivalence of Relation Completeness and Universal Goldbach . 322
Fundamental Theorem AR Readout–Terminal Placement 322
Conclusion What Does a Relation Sieve Actually Sieve? 322
Part V The Exact Architecture of Infinite Relations 326
Abstract 327
Part I Infinite Quantifiers Are Not a Single Concept 333
Chapter 1 From Finite Terminality to Infinite Structure 334
Chapter 2 Seven Principal Global Interfaces 335
2.1 Infinite recurrence 335
2.2 Parameterwise universal fibers 335
2.3 Window universality 335
2.4 Arbitrarily long relations 335
2.5 Parameter infinitude 335
2.6 Indexwise universal predicates 336
2.7 Identity-block separation 336
Chapter 3 The Exact Criterion for Infinite Recurrence 337
Chapter 4 The Exact Criterion for Parameterwise Universal Fibers 338
Chapter 5 Window Universality, Arbitrarily Long Relations, and Parameter Infinitude 339
5.1 Window universality 339
5.2 Arbitrarily long relations 339
5.3 Parameter infinitude 339
Chapter 6 Indexwise Universality and Identity-Block Separation 340
6.1 Indexwise universality 340
6.2 Identity-block separation 340
Chapter 7 Quantifiers Determine the Proof Interface 341
Part II Finite Periodicity and Infinite Persistence 342
Chapter 8 Exact Periodicity at a Finite Control Level 343
Chapter 9 Finite Periodicity Is Not Infinite Relation Periodicity 344
Chapter 10 Three Kinds of Seams 346
10.1 Generation seam 346
10.2 Phase seam 346
10.3 Family of square seams 346
Part III Persistent Non-Extinction Relation Sieves 347
Chapter 11 Why a Fixed Difference Has No Common Terminal Horizon 348
Chapter 12 Unit Difference Is the Minimal Persistent Relation 349
Chapter 13 Matrix-to-Subrelation Inheritance for Unit Difference 350

Chapter 14 Consecutive Prime-Square Horizons 351
Chapter 15 Unique Square-Horizon Assignment 352
Chapter 16 Closure of Square Seams 353
Chapter 17 Old Double-Escape Relations 354
Chapter 18 Single-Layer Purification 355
Chapter 19 Exact Twin Formula in a Square Horizon 356
Chapter 20 Relation Death Leaves a Three-Atom Witness 357
Chapter 21 One Cofactor, One Death 358
Chapter 22 The Clean Zone and the Absorption Zone 359
Chapter 23 The Finite Absorption Resource Pool 360
Chapter 24 Death Matching and Finite Forcing 361
Chapter 25 Local Moving Intervals 362
Chapter 26 Scale Separation Between Relation Space and Resource Space 363
Chapter 27 A Linear Upper Bound for Absorption Resources 364
Chapter 28 Resource Half-Life 365
Chapter 29 Resource Lifetime 366
Chapter 30 Finite Lifetime and Global Capacity Are Two Distinct Levels 367
Part IV Supply–Absorption Theory for General Fixed Patterns 368
Chapter 31 From Twin to an Arbitrary Fixed Offset Pattern 369
Chapter 32 Square Horizons for Patterns 370
Chapter 33 Relation Supply BH 371
Chapter 34 Genuine Relations TH and Deaths DH 372
Chapter 35 The General Form of Single-Layer Purification 373
Chapter 36 The Unified Absorption Resource Pool 375
Chapter 37 The Wheel and the Absorption Reuse Degree 376
Chapter 38 The Absorption-Capacity Theorem 378
Chapter 39 A Purely Structural Upper Bound for the Resource Pool 380
Chapter 40 Exact Capacity and Structural Capacity 381
Chapter 41 Net Relation Potential 382
Chapter 42 Two Types of Finite Relation Certificates 383
42.1 Fixed-spectrum resource certificate 383
42.2 Structural certificate 383
Chapter 43 Several Basic Values of the Absorption Reuse Degree 384
Chapter 44 Twin Is the Minimal Model with $\mu = 1$ 385
Part V From a Single Horizon to Infinite Recurrence 386
Chapter 45 The Cumulative Ledger Across Horizons 387
Chapter 46 Infinite Criteria from Cumulative Net Potential 388
Chapter 47 Stronger Window-by-Window Certificates 390
Chapter 48 Equivalent Form of “There Is No Last Successful Horizon” 391
Chapter 49 The Exact Role of Single-Resource Lifetime in the Cumulative Architecture 392
Chapter 50 Local Admissibility, Finite Net Potential, and Infinite Recurrence Are Three Distinct Conclusions 393
Level 1 Local admissibility 393

Level 2 Finite-horizon certificate 393
Level 3 Infinite recurrence 393
Part VI The Place of Twin in the Exact Infinite Architecture 394
Chapter 51 The Complete Finite State for Twin 395
Chapter 52 How the Global Twin Proposition Is Expressed 396
Chapter 53 General Fixed Differences 397
Chapter 54 General Prime Tuples 398
Part VII Batch Structures for Other Infinite Interfaces 399
Chapter 55 Sophie Germain 400
Chapter 56 Mersenne 401
Chapter 57 Bunyakovsky and Schinzel H 402
Chapter 58 The Arbitrarily-Long Interface of Green–Tao 403
Chapter 59 Goldbach, Legendre, and Universal-Type Interfaces 404
Chapter 60 Brocard, Andrica, Firoozbakht, and Grimm 405
Chapter 61 Erdős–Woods and Identity-Block Separation 406
Part VIII Standard Structures for Batch Proofs 407
Chapter 62 Batch Proof Is Not Batch Computation 408
Chapter 63 Three Standard Closures for Infinite Recurrence 409
63.1 Positive certificate window by window 409
63.2 Cumulative net potential 409
63.3 Direct cumulative lower bound 409
Chapter 64 Three Standard Closures for Universal-Type Propositions 410
64.1 Uniform positive lower bound 410
64.2 Finite-quotient coverage 410
64.3 Exclusion of the least failure 410
Chapter 65 Standard Closure for Arbitrarily Long Relations 411
Chapter 66 Standard Closure for Parameter Infinitude 412
Part IX Finite Certificates at the Relation Level 413
Chapter 67 Relation Objects and Certificate Objects Should Have the Same Granularity 414
Chapter 68 General Single-Relation Certificate for Twin 415
Chapter 69 Certificate from One Sufficiently Thick Principal Chain 417
Chapter 70 Moving Horizons and Relation Certificates Are the Same Principle at Two Scales 419
Part X Finite Computational Verification 420
Chapter 71 905 Consecutive Square Horizons 421
Chapter 72 Structural Net Potential on the Same Finite Data 422
Chapter 73 Finite Verification of Historical Resource Reuse 423
Chapter 74 Computational Comparison of Relation-Level Certificates 424
Part XI Parameter Dictionary 426
Chapter 75 Meaning of the Parameters in the Infinite-Relation Module 427
75.1 H: fixed offset pattern 427
75.2 $p < r < s$: consecutive odd primes 427
75.3 $\Delta = s - r$: current prime gap 427
75.4 BH: relation supply 427

75.5 TH: genuine relations 427
75.6 DH: genuine deaths 427
75.7 $U(r, s)$: absorption resource pool 427
75.8 Q: set of wheel control primes 427
75.9 MQ: wheel modulus 428
75.10 $\mu H(Q)$: absorption reuse degree 428
75.11 CHexact: fixed-spectrum resource capacity 428
75.12 CHstruct: structural capacity 428
75.13 eH: net relation potential 428
75.14 $\mathcal{E}H$: cumulative net relation potential 428
Chapter 76 Meaning of the Parameters in the Quantifier Module 429
X 429
θ 429
x 429
k 429
$CR(X)$ 429
$C(\theta)$ 429
$W(n)$ 429
$L(X)$ 429
$P(X)$ 429
$F(X)$ 429
Part XII Finite Certificates Enter Formal Proofs of Classical Propositions 430
Chapter 77 Certificates as Theorem Interfaces 431
77.2 Certificate transfer for infinite recurrence 431
77.3 Certificate entries for the other quantifiers 432
Chapter 78 Formal Proof and Infinite Closure of Twin 433
Chapter 79 How General Classical Propositions Invoke Part V 435
79.1 Fixed prime tuple 435
79.2 Goldbach-type universal fibers 435
79.3 Window universality 435
79.4 Arbitrary length 435
79.5 Parameter infinitude 436
79.6 Unified invocation chain 436
Part XIII Table of Core Theorems in This Part 437
Chapter 80 Infinite-Quantifier Theorems 438
T1 Counting Criterion for Infinite Recurrence 438
T2 Universal-Fiber Criterion 438
T3 Window-Universality Criterion 438
T4 Arbitrary-Length Criterion 438
T5 Parameter-Infinitude Criterion 438
Chapter 81 Persistent Relation-Sieve Theorems 439
T6 Exact Twin Formula in a Square Horizon 439
T7 One Cofactor, One Death 439

T8 Clean Zone 439
T9 Resource Half-Life 439
T10 Finite Resource Lifetime 439
Chapter 82 Supply–Absorption Theorems for General Patterns 440
T11 Single-Layer Purification 440
T12 Absorption Capacity 440
T13 Structural Resource Upper Bound 440
T14 Exact Net-Potential Lower Bound 440
T15 Structural Net-Potential Lower Bound 440
T16 Infinite Criterion from Cumulative Exact Net Potential 440
T17 Infinite Criterion from Cumulative Structural Net Potential 440
Chapter 83 Relation-Level Certificate Theorems 441
T18 General Single-Relation Twin Certificate 441
T19 Certificate from One Sufficiently Thick Principal Chain 441
T20 Certificate Transfer for Universal Fibers 441
T21 Certificate Transfer for Infinite Recurrence 441
T22 Twin Certificate from Cumulative Net Potential 442
T23 Certificate Interface for Classical Propositions 442
Conclusion Infinity Is the Unbounded Accumulation of Finite Certificates 442
Part VI Formal Proofs of Classical Propositions, the Complete Family of Relation Slices, and Method Boundaries 446
Abstract 447
Part I The Carrying Form of the Unified Matrix 453
Chapter 1 The Relation Matrix Precedes Named Problems 454
Chapter 2 The Five Layers of the Unified Matrix 455
2.1 Fixed-identity layer 455
2.2 Atomic layer 455
2.3 Relation layer 455
2.4 Horizon layer 455
2.5 Readout layer 455
Chapter 3 Arbitrary Finite-Order Relation Spaces 457
Chapter 4 Multiplicative-Depth Tensor and Atomic Vertex Skeleton 458
Chapter 5 Vertices, Edges, Faces, and Shallow Multiplicative Depth 459
Chapter 6 Relation Orbits 460
Chapter 7 Atomic-Vertex Preimage Theorem 461
Chapter 8 Unified Quadruple Representation of a Relation 462
8.1 Relation orbit ΓF 462
8.2 Target depth B 462
8.3 Observation region $\mathcal{R}$ 462
8.4 Global requirement Q 462
Chapter 9 Terminal Certification of Finite Relations 464
Chapter 10 How Finite Worlds Compose a Global Relation 465
Part II Six Geometries of Relations 467

Chapter 11 Fixed Relation Slices 468
Chapter 12 Higher-Order Affine Relations 469
Chapter 13 Nonlinear Relation Orbits 470
Chapter 14 Sparse Exponential Relation Orbits 471
Chapter 15 Interval and Metric Relations 472
Chapter 16 Statistical and Spectral Relations 473
Part III Formal Proofs of Classical Propositions and the Complete Family of Relation Slices 474
Formal-Proof Convention for This Part 474
Chapter 17 Goldbach: Universal Fibers of Fixed Sum 475
17.A Formal Equivalence Theorem for Goldbach 475
Chapter 18 Lemoine / Lévy: Weighted Reflection Fibers 477
18.A Formal Equivalence Theorem for Lemoine / Lévy 477
Chapter 19 Weak Goldbach: Raising the Relation Order to Three 479
19.A Ternary Formal Proof of Weak Goldbach 479
Chapter 20 Chen: Same Relation Geometry, Relaxed Target Depth 480
20.A Formal Proof of a Chen-Type Proposition 480
Chapter 21 Twin: Infinite Recurrence of Unit Fixed Difference 481
21.A Formal Equivalence and Certificate Chain for Twin 481
Chapter 22 Polignac: The Entire Family of Fixed Differences 483
22.A Formal Proof for Polignac 483
Chapter 23 Sophie Germain and Safe Primes: A Slanted Relation 485
23.A Formal Proof for Sophie Germain / Safe Primes 485
Chapter 24 Hardy–Littlewood Prime k-Tuples 486
24.A Formal Proof for the Hardy–Littlewood Prime k-Tuple 486
Chapter 25 Green–Tao: Arbitrarily Long Arithmetic-Progression Relations 487
25.A Formal Proof for Green–Tao 487
Chapter 26 Mirror Folding from Green–Tao to a Fixed Sum 488
26.A Mirror-Folding Theorem 488
Chapter 27 Dirichlet: Infinite Occupancy on Congruence Sublattices 489
27.A Relational Form of Dirichlet's Theorem 489
Chapter 28 PNT and PNT in AP: Coarse Density 490
28.A Statistical Forms of PNT and PNT in AP 490
Chapter 29 Bertrand: Occupancy of a Moving Interval 491
29.A Window-Form Proof of Bertrand's Postulate 491
Chapter 30 Legendre: Windows Between Consecutive Integer Squares 492
30.A Window-Form Proof of Legendre's Conjecture 492
Chapter 31 Oppermann: Universality of Two Windows 493
31.A Two-Window Formal Proof of Oppermann's Conjecture 493
Chapter 32 Brocard: Counting Primes Between Consecutive Prime Squares 494
32.A Square-Window Counting Form of Brocard's Conjecture 494
Chapter 33 Andrica: A Metric Relation for Prime Gaps 495
33.A Equivalent Metric Form of Andrica's Conjecture 495
Chapter 34 Firoozbakht: Normalized Growth of Consecutive Primes 496

34.A Equivalent Logarithmic Form of Firoozbakht's Conjecture 496
Chapter 35 $n2 + 1$: A Quadratic Orbit 497
35.A Formal Proof for $n2 + 1$ 497
Chapter 36 Bunyakovsky: General Polynomial Orbits 498
36.A Formal Proof for Bunyakovsky 498
Chapter 37 Schinzel H: Polynomial Families 499
37.A Formal Proof for Schinzel H 499
Chapter 38 Mersenne: A Sparse Exponential Orbit 500
38.A Formal Proof for Mersenne Primes 500
Chapter 39 Fermat: A Doubly Exponential Sparse Orbit 501
39.A Formal Proof for the Fermat-Prime Orbit 501
Chapter 40 Waring–Goldbach: A Nonlinear Fixed-Sum Manifold 502
40.A Formal Proof of a Waring–Goldbach-Type Proposition 502
Chapter 41 Grimm: Complete Factor Sets and Matching 503
41.A Matching-Form Proof of Grimm's Conjecture 503
Chapter 42 Erdős–Woods: Identity-Block Separation 504
42.A Identity-Block Formal Proof for Erdős–Woods 504
Chapter 43 Four Different Readouts of Landau's Four Problems 505
43.A Unified Formal Table for Landau's Four Problems 505
Chapter 44 RH / GRH: The Global Spectral Interface 507
44.A Global Spectral Form of RH / GRH 507
Part IV Convolution Histories, Terminal Factorization, and Derived Structures 508
Convolution Exponential: One Parameter Encodes All Finite Relation Orders 508
Finite Terminal Convolution Factorization 508
Zero Mean on Coarse–Fine Fibers 509
Seven Derived Relation Structures 509
Part V How Higher-Order Relations Transfer Information to Lower-Order Relations 512
Chapter 45 Structural Maps Between Relations 513
Chapter 46 Lossless Higher-Order-to-Lower-Order Transfer 514
Chapter 47 Reverse Readout from the Fixed-Sum Witness Field to Prime Constellations 515
Chapter 48 Higher-Order Existence Does Not Imply Lower-Order Universality 516
Chapter 49 Why Lower-Order Relations Have the Least Redundancy 517
Chapter 50 Four Relaxations Rejected by Lower-Order Propositions 518
50.1 Increasing the number of variables 518
50.2 Relaxing multiplicative depth 518
50.3 Relaxing the relation fiber 518
50.4 Allowing an exceptional set 518
Chapter 51 Structure May Be Lifted in Order; Witnesses Must Not Be Lost 519
Part VI The Place of Classical Sieve Methods in the Unified Matrix 520
Chapter 52 The Sieve of Eratosthenes: Multiplicative Occupancy 521
Chapter 53 Brun's Sieve: Multi-Endpoint Rough Relations 522
Chapter 54 Selberg's Sieve: Optimizing Local Exclusion Does Not Add Missing Identity Information 523

Chapter 55 Combinatorial and Linear Sieves: Shallow Multiplicative Depth 524
Chapter 56 The Large Sieve: Multi-Modulus Average Control 525
Chapter 57 Bombieri–Vinogradov: Average Distribution Across Moduli 526
Chapter 58 The Hardy–Littlewood Circle Method: Frequency Readout for Fixed Sums 527
Chapter 59 Why Ternary and Higher-Order Additive Relations Have More Smoothing 528
Chapter 60 Chen: Relaxing Depth While Keeping the Relation Fixed 529
Chapter 61 GPY and Maynard–Tao: Differences Compressed into a Finite Band 530
Chapter 62 Green–Tao: Higher-Order Combinatorial Structure 531
Chapter 63 Analytic *L*-Functions: The Global Spectrum 532
Chapter 64 Pocklington, Lucas, APR, and ECPP: Finite Certificates 533
Part VII Primorial Coarse–Fine Decomposition and Analytic Relation Sieves 534
Chapter 65 The Coarse Space on a Finite Cyclic Group 535
Chapter 66 Strict Separation of Fourier Supports 536
Chapter 67 Exact Decoupling Under Convolution 537
Chapter 68 Primorial Shells 538
Chapter 69 Forced Sieve Direction and Transverse Innovation 539
Chapter 70 Recursion for Coarse Energy 540
Chapter 71 Transport of Local Factors 541
Chapter 72 Finite Duhamel Formula 542
Part VIII Parity, Child Modes, and the Boundary of Binary Relations 543
Chapter 73 Parity of Multiplicative Depth 544
Chapter 74 Rough Survivors and Signed Survivors 545
Chapter 75 Parity-Resolved Decomposition of a Fixed Sum 546
Chapter 76 The Absorbing Nature of Parity Saturation 547
Chapter 77 First Extinction and Vertex Covers of the Relation Graph 548
Chapter 78 Ordinary Death and True Death 549
Chapter 79 Child-Mode Anisotropy 550
Chapter 80 Analytic Duality Between Goldbach and Fixed Differences 552
Chapter 81 The Distance Between Average Control and All-Target Control 553
Chapter 82 The Information Boundary of Pure Positive Energy 554
Chapter 83 Boundary of a Single-Scale Reflection Gap 555
Chapter 84 Moving-Target Determinant Structure 556
Chapter 85 RWF-II as an Analytic Interface 557
Part IX Master Table of Method Capability Boundaries 558
Chapter 86 Methods and Information Levels 559
Chapter 87 When Can a Higher-Order Method Enter a Lower-Order Proposition? 561
Chapter 88 Almost-All and All-Target 562
Chapter 89 Bounded Band and Fixed Fiber 563
Chapter 90 Almost-Primes and Prime Depth 564
Chapter 91 Higher-Order Smoothing and Lower-Order Rigidity 565
Part X Generative Complexity 566
Chapter 92 How Should the Difficulty of Relations Be Compared? 567
Chapter 93 Atomic-Layer Complexity 568

Chapter 94 Horizon Complexity 569
Chapter 95 Relation Complexity 570
Chapter 96 Certificate Complexity 571
Part XI Unified Master Tables of Classical Relations 572
Chapter 97 Master Table of Relation Positions 573
Chapter 98 Master Table of Landau's Four Problems 574
Chapter 99 Standard Procedure for Entering a New Relation into the Matrix 575
Chapter 100 Unify the Object Layer, Not the Concrete Tools 577
Master Table of Formal Proofs of Classical Propositions 577
Conclusion Classical Problems Are No Longer Isolated, and Classical Methods Are No Longer in Conflict 579
Part VII RenSpectrum: Executable Implementation of the Theory and an Open Relational Language 583
Abstract 584
Part I Name, Positioning, and Execution Boundaries 589
Chapter 1 RenSpectrum: The Full Meaning of the Name 590
1.1 Ren 590
1.2 Spectrum 590
1.3 Prime-Relation 591
1.4 Workbench 591
Chapter 2 The Visual Identity Is Also Part of the Relational Language 592
Chapter 3 The Software Is Not the Source of Proof, but an Execution Environment for Proof Semantics 593
Chapter 4 The Seven Execution Layers of RenSpectrum 594
Part II Rapid Relation Readout 595
Chapter 5 Unified Compilation Formula for Rapid Readout 596
Chapter 6 Anchor: The Free Position of a Relation Instance 597
Chapter 7 Input Fields for Rapid Readout 598
7.1 Relation Type 598
7.2 Parameter 1 / Parameter 2 / Parameter 3 598
7.3 Readout Horizon 598
7.4 Control Depth z 598
7.5 Display Filtering 598
Chapter 8 The Itemwise Relation Spectrum 600
Chapter 9 Fixed Sums and Weighted Reflection 601
Chapter 10 Fixed Differences and Consecutive Fixed Gaps 602
Chapter 11 Affine Relations 603
Chapter 12 Fixed-Offset Patterns and a Blocking Example 604
Chapter 13 Fixed Green–Tao Slices 605
Chapter 14 Window Relations 606
14.1 Legendre 606
14.2 Oppermann 606
14.3 Brocard 606

Chapter 15 Neighboring-Prime Indices 607
Chapter 16 Exponential and Polynomial Pullbacks 608
16.1 Mersenne 608
16.2 $n2 + 1$ 608
16.3 Bunyakovsky 608
16.4 Schinzel H 608
Chapter 17 Relations Using Complete Factor Identities 609
17.1 Grimm 609
17.2 Erdős–Woods 609
Chapter 18 Three Examples of Mixed Relations 610
18.1 Cross-Mirror Relation 610
18.2 Quadratic Double-Atom Relation 610
18.3 Reflection–Translation Coupling 610
Part III Infinitude and Universality Modules 612
Chapter 19 The Software First Identifies the Quantifier Type 613
Chapter 20 INF_RECURRENCE 614
Chapter 21 UNIVERSAL_FIBER 615
Chapter 22 WINDOW_UNIVERSAL 616
Chapter 23 UNIVERSAL_PREDICATE 617
Chapter 24 ARBITRARILY_LONG 618
Chapter 25 PARAMETER_INFINITE 619
Chapter 26 BLOCK_SEPARATION 620
Part IV Seams, Cycles, and Persistence 621
Chapter 27 Executable Definitions of Three Types of Seams 622
27.1 Generation Seam 622
27.2 Phase Seam 622
27.3 Square-Seam Family 622
Chapter 28 Finite Cycle $S0$ and Global Persistence $S\infty$ 623
Chapter 29 Square-Seam Closure Scan 624
Chapter 30 Persistent Readout Across Square Windows 625
Part V Local Obstruction and the Emission–Absorption Executor 626
Chapter 31 Normalization of a Pattern H 627
Chapter 32 Local-Obstruction Fingerprint 628
Chapter 33 Admissible Status of a Pattern 629
Chapter 34 BH: Relation Emission 630
Chapter 35 U: Absorption Resources 631
Chapter 36 Q and $\mu H(Q)$ 632
Chapter 37 Two Absorption Capacities 633
Chapter 38 Two Net Relation Potentials 634
Chapter 39 TH and DH 635
Chapter 40 Which Relations Enter the Balance Module Automatically? 636
Part VI RSL: The Open Relational Language 637
Chapter 41 Name and Purpose of RSL 638

Chapter 42 The Matrix–Subrelation Discipline of RSL 639
Chapter 43 Basic Structure of RSL 640
Chapter 44 The `anchor` Field 641
Chapter 45 The `parameters` Field 642
Chapter 46 `maps` 643
Chapter 47 `map_generator` 644
Chapter 48 `predicate` 645
Chapter 49 `quantifier` 646
Chapter 50 RSL Expression of Twin 647
Chapter 51 RSL Expression of Goldbach 648
Chapter 52 RSL Expression of Polignac 649
Chapter 53 RSL Expression of a Fixed Green–Tao Slice 650
Chapter 54 A Mixed Polynomial Relation 651
Chapter 55 The RSL Compilation Pipeline 653
55.1 Executable Output of the Formal-Proof Chain 653
55.2 A Relation Name Is Not Execution Semantics 654
Chapter 56 The Open Principle of RSL 655
Part VII Computational Budget and Execution Backends 656
Chapter 57 Why the Computational Budget Must Be Estimated First 657
Chapter 58 Memory Model of the Web Dense Spectrum 658
Chapter 59 Four Task-Risk Levels 659
Chapter 60 Web Engineering Thresholds 660
Chapter 61 Adaptive Device-Memory Tiers 661
Chapter 62 CPU Threads and Browser Capability Detection 662
Chapter 63 JavaScript Integer Boundary 663
Chapter 64 Mersenne and BigInt 664
Chapter 65 High Coordinates Do Not Imply Large Memory 665
Chapter 66 Segmented Sieve 666
Chapter 67 Web Worker 667
Chapter 68 WebAssembly 668
Chapter 69 Wasm Multithreading 669
Chapter 70 The Proper Role of WebGPU 670
Chapter 71 The GPU Should Not Carry Every Task 671
Chapter 72 RenSpectrum Desktop 672
Chapter 73 Four Execution Backends 673
Chapter 74 Relation Types and Recommended Backends 674
Chapter 75 A Reasonable Performance Roadmap 675
Part VIII Verification Levels 676
Chapter 76 Why Verification Levels Are Necessary 677
Chapter 77 `V0`: Syntax Is Executable 678
Chapter 78 `V1`: Identity Readout Is Independently Checkable 679
Chapter 79 `V2`: Finite Relation Readout Is Independently Checkable 680
Chapter 80 `V3`: Finite Terminal Certificate 681

Chapter 81 V4: Structural Batch Certificate 682
Chapter 82 V5: Global Quantifier Closure 683
Chapter 83 Basic Discipline of Verification Levels 684
83.1 Verification Levels and the Formal-Proof Chain 684
Part IX Machine Checking and Reproducible Experiments 686
Chapter 84 Determinism First 687
Chapter 85 Minimum Record for Every Run 688
Chapter 86 Software Version and Relation Version Must Both Be Preserved 689
Chapter 87 Completeness Checks for Finite Output 690
Chapter 88 Blocking Examples Are Foundational Regression Tests 691
Chapter 89 Relation Identities as Cross-Checkers 692
Chapter 90 Ledger Check for $BH = TH + DH$ 693
Chapter 91 The Overlap Region of the BigInt and Number Paths 694
Chapter 92 Semantic Consistency Between Web and Desktop 695
Part X Complete Parameter Dictionary 696
Chapter 93 Identity-Spectrum Parameters 697
$\boldsymbol{I}(n)$ 697
$\alpha(n)$ 697
$P-(n)$ 697
$\Omega(n)$ 697
$\mathrm{PF}(n)$ 697
A 697
$\mathcal{M}$ 697
Chapter 94 Relational-Language Parameters 698
Θ 698
x / anchor 698
fi 698
$\mathcal{B}$ 698
Q 698
H 698
k 698
d 698
N 698
A, B 698
Chapter 95 Finite-Horizon and Seam Parameters 699
z 699
Wz 699
$p < r < s$ 699
$r2$ 699
$s2$ 699
Δ 699
Chapter 96 Obstruction Parameters 700
$vH(q)$ 700

$vH(q)$ 700
admissible 700
Chapter 97 Supply–Absorption Parameters 701
BH 701
$U(r, s)$ 701
Q 701
MQ 701
$JH(c; Q)$ 701
$\mu H(Q)$ 701
CHexact 701
CHstruct 701
eHexact 701
eHstruct 701
TH 701
DH 702
Chapter 98 Quantifier Parameters 703
$CR(X)$ 703
$C(\theta)$ 703
$W(n)$ 703
$L(X)$ 703
$P(X)$ 703
$F(X)$ 703
$K*$ 703
Chapter 99 Software-Budget Parameters 704
max coordinate 704
anchors 704
dense bytes 704
CPU logical threads 704
browser-visible memory tier 704
WebGPU 704
cross-origin isolation 704
DOM row cap 704
task risk 704
Chapter 100 RSL Field Dictionary 705
Part XI Reproducible Experiment Protocols 706
Chapter 101 Goal of the Protocols 707
Chapter 102 Protocol A: Fixed-Relation Readout 708
Chapter 103 Protocol B: Obstruction Certificate 709
Chapter 104 Protocol C: Emission–Absorption Certificate 710
Chapter 105 Protocol D: Cumulative Infinitude Interface 711
Chapter 106 Protocol E: BigInt and Large-Integer Certificates 712
Chapter 107 Protocol F: Web / Desktop Cross-Verification 713
Chapter 108 Protocol G: Version Archiving 714

Part XII Open Architecture 715
Chapter 109 Preset Relations Are Not the Boundary of the System 716
Chapter 110 From a “Proof Problem” to a “Compilation Problem” 717
Chapter 111 What Is Unified Is Not Notation but Information Flow 718
Chapter 112 Why Problems Take the Forms They Do 719
Chapter 113 Openness Does Not Mean Adding Infinitely Many Buttons 720
Chapter 114 New Relations Can Also Test the Language of the Theory 721
Chapter 115 The Software Does Not Think for the Reader 722
Chapter 116 An Open Workbench Should Preserve Failure 723
Chapter 117 From the Fixed Spectrum to Open Knowledge Generation 724
Part XIII Closing the Book 726
Chapter 118 The Seven Parts Seen Again as One Whole 727
Chapter 119 Atoms Before Problems, Relations Before Names 728
Chapter 120 From “How Can This Be Proved?” to “Why Does the Problem Appear in This Form?” 729
Chapter 121 Classical Methods Are No Longer Isolated from One Another 730
Chapter 122 The Unified Matrix Is Not an Archive of Knowledge 731
Chapter 123 The True Meaning of Openness: A Common Entry Point and Freedom of Method 732
123.1 Different People Can Enter Through Different Doors 732
123.2 The Mathematical Pleasure of Open Work 733
Chapter 124 Division of Labor Between Machines and People 734
Chapter 125 How a Reader Can Continue Using This Book 735
Chapter 126 From the Web Page to an Open Implementation 736
Chapter 127 The Final Parameters of the Book Are No Longer Merely Parameters 738
Chapter 128 Final Conclusion of the Book 740
How to Use the References in This Book 742
References 743
I. Classical Foundations, the History of Primes, and the Beginnings of Analytic Number Theory 743
II. Goldbach, Additive Prime Problems, and the Circle Method 743
III. Sieve Methods, Local Obstruction, Almost-Primes, and Distributional Tools 744
IV. Small Gaps, Twin Primes, Prime Tuples, and Arithmetic-Progression Structure 744
V. Classical Prime Relations, Windows, Polynomials, and Factor Identities 744
VI. Global Spectral Theory, the Prime Number Theorem, and Multiplicative Number Theory 745
VII. Finite Primality Certificates, Algorithms, and Executable Verification 745
VIII. General Texts, History, and Further Reading 745
Guide to the Use of References 746
Postscript: After Unification, a New Journey Begins 747
I. The Seven Parts Are Not Seven Theories, but Seven Layers of One Object 747
II. Goldbach and Twin: The Simultaneous Placement of the Two Most Fundamental Global Relations 748
III. The Real Meaning of “Solving in One Sweep”: No Longer Rebuilding a World for Every Conjecture 748

IV. After Proof, the Truly Vast Work Moves to the Back End 749
V. Every Era Has Its Own Mission 751
VI. From a “Catalog of Prime Conjectures” to a “Science of Prime Relations” 751
VII. After Unification, the Distance Becomes Clear for the First Time 752
Every Prime Is Lonely, Yet No Prime Is Truly Isolated 755

Preface: From Isolated Prime Mysteries to a Unified World of Relations

I. Prime Numbers: Among the Oldest Objects in Mathematics, Yet Among Its Youngest Problems

Across the long history of mathematics, few objects are as ancient as prime numbers and yet remain so young as problems.

They are ancient because divisibility, factors, parity, and integers that are "no longer decomposable" entered human mathematical experience almost as early as counting itself. Ancient Greek mathematics elevated this experience into rigorous proof. Proposition 20 of Book IX of Euclid's *Elements* proves that there is no last prime: no matter how many primes are given, one can infer the existence of another. [1] This is a quintessential mathematical miracle—an infinite fact compressed into a finite argument.

The Sieve of Eratosthenes displays another enduring idea: one need not know all primes in advance; one may instead let known multiplicative structure cover reducible positions layer by layer, so that the candidates left uncovered become visible. More than two millennia later, sieve methods remain one of the central languages of prime-number theory. The techniques have changed, but the ancient structural line has not: **multiplication determines which positions can be explained, while the remaining positions reveal atomic identity.**

Prime numbers are also young because, even today, many questions that can be stated using only elementary arithmetic still mobilize some of the deepest tools in number theory: can every even integer always be decomposed into two primes? Do pairs of primes differing by 2 occur infinitely often? Is there always a prime between consecutive squares? Given several fixed offsets, do they simultaneously land on primes infinitely often? Can a polynomial orbit keep hitting primes?

These problems look very different on the surface, yet all move us from the question "what is one integer?" to the question "how do several integers jointly form a relation?"

Prime numbers have never lacked a definition.

What is often truly difficult is the **relation**.

If one asks only whether an integer is prime, the answer is determined by that integer's own multiplicative identity. Once one asks whether two, three, or more positions can simultaneously satisfy a sum relation, difference relation, window condition, affine relation, arithmetic-progression pattern, polynomial relation, or adjacency condition, the problem moves from identity to relation. Add quantifiers such as "for every," "there exist infinitely many," "arbitrarily long," or "every window contains one," and a local relation immediately becomes a global structure.

This book begins from precisely this distinction.

Its question is not whether one can invent yet another technique for one classical conjecture, but rather a more foundational question:

Do the many prime-number problems historically named and studied separately actually belong to one and the same relation matrix?

If the answer is yes, then Goldbach, Twin, Polignac, Legendre, Oppermann, Brocard, Sophie Germain, prime k-tuple problems, Green–Tao, prime values of polynomials, Mersenne, Grimm, Erdős–Woods, and others should not first be regarded as a long list of unrelated conjecture names.

They should first be returned to the same fixed arithmetic identity spectrum, the same atomic relation matrix, the same finite-certification structure, and the same quantifier language.

That is the starting point of *Unified Prime Relations Theory*.

II. From Euclid and Euler to Riemann: Prime Numbers Move from "Atoms" to Global Structure

Euclid tells us that prime numbers do not terminate; Eratosthenes tells us that primes can be found by excluding multiplicatively occupied positions. By the eighteenth century, Euler had connected primes deeply with infinite series and infinite products. Euler products link unique prime factorization of integers with ζ-type series, so that the idea of "primes as multiplicative atoms" ceases to be merely a discrete classification and enters analytic structure. [2][4]

In the nineteenth century, prime-number theory moved further from "one prime after another" toward the question "how are primes distributed as a whole?" Gauss and Legendre developed systematic conjectures for the growth of the prime-counting function; in 1837 Dirichlet proved that every arithmetic progression in a reduced residue class contains infinitely many primes, thereby connecting prime existence with L-functions, characters, and congruence structure. [5]

In 1859, Riemann published the few pages of *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, a paper that profoundly redirected number theory. There, the distribution of primes entered an entirely new spectral world through the complex zeros of $\zeta(s)$. [6] In 1896, Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem, confirming

$$\pi(x) \sim \frac{x}{\log x},$$

and thereby raising the distributional intuition of Gauss and Legendre to the status of theorem. [7][8]

This historical line is important. It shows that prime-number research has never advanced along a single technical route:

irreducibility → sieving → series and products → congruence distribution → complex analysis and spectrum

Each major advance did not simply discard older objects; rather, it found a representation capable of carrying more of the global structure.

This book continues that tradition, but pushes the question one step further:

Once the pointwise identity of primes is fixed, can one build an equally unified underlying object for **relations among primes**?

III. Goldbach: From Prime Identity to Fixed-Sum Relations

In 1742, correspondence between Christian Goldbach and Leonhard Euler left us the famous problem that later took Goldbach's name. [3] Because eighteenth-century usage regarding whether 1 counted as prime differed from modern convention, the original formulation is not identical to today's version. The modern binary Goldbach proposition is usually written as

Every even integer greater than 2 can be expressed as a sum of two primes.

Viewed only as a formula, this is simply

$$N = p + q.$$

Yet this elementary equation pushes prime-number research sharply onto the level of relations.

For a fixed even integer N, the question is not merely how many primes lie in a range. It requires the existence of two exactly corresponding positions

$$p, \qquad N - p,$$

both carrying prime identity.

Thus the difficulty of Goldbach has never been equivalent to the question "are there enough primes?" What it requires is a **witness inside a fixed reflection relation**.

Two intervals may contain comparable numbers of primes without forcing a prescribed sum to produce a pair; abundance on average does not automatically imply that every target fiber is nonempty. Here, for the first time, the levels of "identity" and "relation" become visibly non-interchangeable.

Goldbach then became one of the central axes of additive number theory. The Hardy–Littlewood circle method and singular series placed it inside an analytic framework; in 1937 Vinogradov proved that every sufficiently large odd integer is a sum of three primes; Chen Jingrun pushed the binary problem to "a prime plus a product of at most two primes"; Helfgott ultimately completed the weak Goldbach theorem, showing that every odd integer greater than 5 is a sum of three primes. [11][12][13][14]

These results repeatedly approached the binary statement, while making increasingly clear one structural fact:

high-order averaging, ternary smoothing, almost-primes, and binary targetwise universality do not carry the same information.

Goldbach demands the binary relation itself, with minimal redundancy.

IV. Twin Primes and Polignac: The Other Fundamental Axis Is Fixed Difference

If Goldbach is a fixed-sum relation, then the twin-prime problem takes us to the other fundamental axis of a two-atom relation: fixed difference.

$$p, \qquad p + 2$$

Are both prime infinitely often?

In 1849, Alphonse de Polignac proposed the more general fixed-even-gap problem: every positive even integer should occur infinitely often as the difference between consecutive primes. [9] Twin primes are the special case of the smallest nontrivial even gap, 2.

This time the outer quantifier is entirely different from Goldbach's.

Goldbach requires

$$\forall N \quad \exists (p, q),$$

that is, **every target must be hit**.

Twin requires

$$\exists^{\infty} p,$$

that is, **the same fixed relation must keep returning**.

Both propositions involve only two primes, yet they represent two fundamentally different global structures:

parameterwise universal nonemptiness versus infinite recurrence of a fixed relation.

At the beginning of the twentieth century, Viggo Brun developed new sieve methods and proved convergence of the sum of reciprocals of twin primes. [17] This is both profound and revealing: it rigorously controls the sparsity of twin primes, yet does not decide whether there are infinitely many of them. In the twenty-first century, Goldston–Pintz–Yıldırım developed a new framework for small prime gaps; in 2013 Yitang Zhang proved that infinitely many consecutive prime pairs have gap less than seventy million; Maynard subsequently used a new multidimensional sieve-weight method to strengthen bounded-gap results to richer multiprime forms. [23][24][25]

These works changed the scale of prime-gap research, yet the simplest fixed relation remains at the very front:

$$\boxed{q - p = 2.}$$

Between a bounded band of gaps and one prescribed fixed-gap fiber lies a structural distinction that cannot be replaced by saying that the two are "very close."

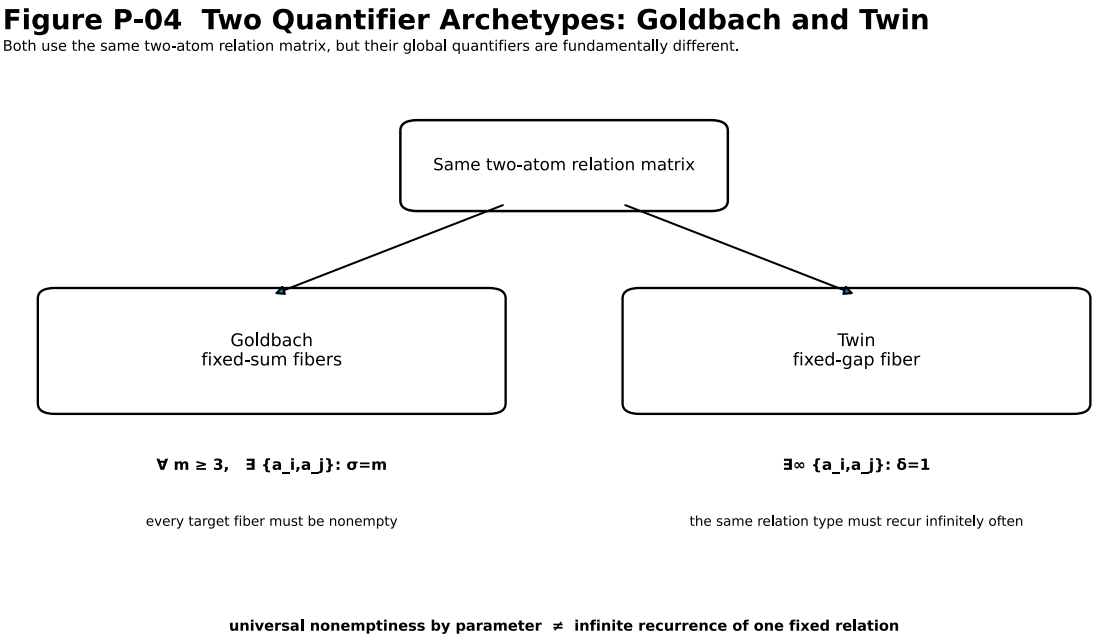


Figure P-04. Two Quantifier Archetypes: Goldbach and Twin. Goldbach and Twin are both two-atom relations, but their outer quantifiers differ: Goldbach requires every fixed-sum target fiber to be nonempty, while Twin requires one fixed-gap relation to recur infinitely often. Universal nonemptiness by parameter and infinite recurrence of a fixed relation are not interchangeable.

V. Landau's Four Problems: A Concentrated Naming of Prime-Number Difficulties at the Beginning of the Twentieth Century

In 1912, at the International Congress of Mathematicians in Cambridge, Edmund Landau listed four famous prime-number problems and described them as unassailable in the state of science at that time. Later literature usually refers to them as Landau's four problems: [10]

1. Binary Goldbach;
2. Twin primes;
3. Whether there is always a prime between two consecutive squares;
4. Whether there are infinitely many primes of the form $n^2 + 1$.

Placed side by side, these four problems almost form a natural "classification chart of relations."

Goldbach is fixed-sum reflection; Twin is fixed-difference translation; the Legendre-type problem is occupancy of a moving window; $n^2 + 1$ is prime hitting along a nonlinear polynomial orbit.

Traditional methods may of course treat them separately, but in the language of this book they can all first be written in the same basic form:

fixed identity spectrum + position map + local identity condition + global quantifier.

This does not mean that the four problems are "the same."

On the contrary, the value of a unified language is that the points at which they are genuinely different can, for the first time, be compared precisely.

Unification does not erase differences.

Unification makes differences appear in the same coordinate system.

VI. The Twentieth and Twenty-First Centuries: Great Advances Keep Compressing the Problems Toward Their Lowest-Order Core

Twentieth-century prime-number theory created an extraordinarily powerful toolbox.

The Hardy–Littlewood circle method expressed fixed-sum problems through Fourier superposition and singular series; Vinogradov's exponential-sum techniques established ternary representations of sufficiently large odd integers; Selberg developed widely applicable sieve methods; the Bombieri–Vinogradov theorem pushed prime distribution in arithmetic progressions to powerful average-modulus control; Chen advanced Goldbach to $P_1 + P_2$; the large sieve, linear sieve, and combinatorial sieve continually strengthened control of rough numbers and almost-primes. [11]–[22]

In the twenty-first century, the GPY, Zhang, and Maynard–Tao lines brought prime-gap questions into finite bands; Green and Tao proved that the primes contain arbitrarily long arithmetic progressions; Helfgott completed ternary Goldbach. [14][23]–[27]

Together these achievements suggest an increasingly clear lesson:

> Number theory already possesses extremely strong forms of "global control," while some of its oldest propositions demand the least redundant, most concrete, and most rigid relational witness.

Binary Goldbach requires every fixed-sum slice to contain an actual prime–prime witness; Twin requires the unit fixed-gap fiber itself to extend without bound.

Thus historical progress has not made "relation" less important. On the contrary, it has repeatedly compressed what remains difficult down to the relation itself.

That is the point of entry for this book's reorganization of the subject.

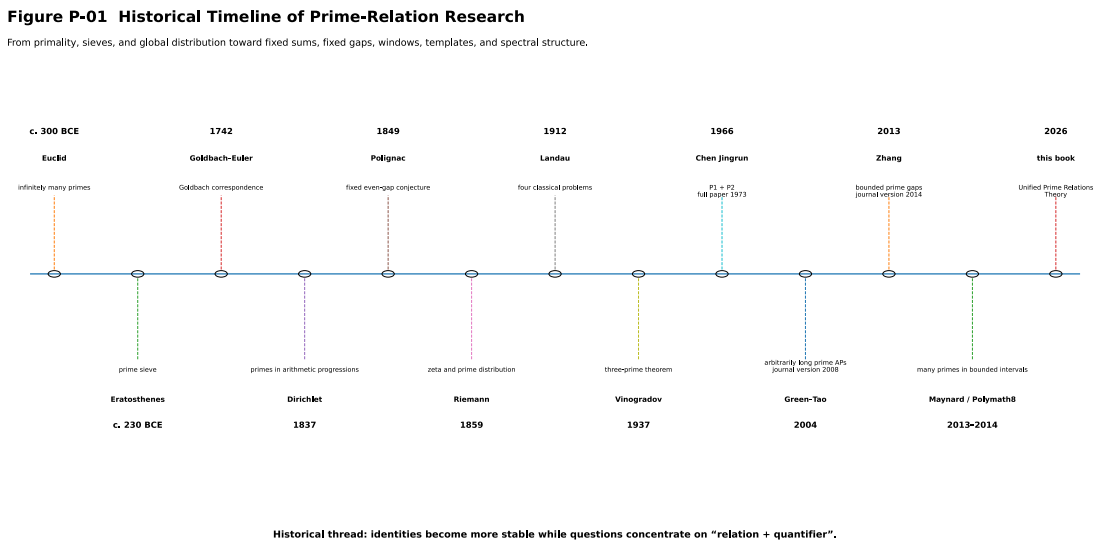


Figure P-01. Historical Timeline of Prime-Relation Research. From Euclid’s infinitude of primes and the sieve of Eratosthenes through Goldbach–Euler fixed sums, Dirichlet progressions, Polignac fixed gaps, and Riemann’s spectral viewpoint, then Landau, Vinogradov, Chen, Green–Tao, Zhang, and Maynard, the historical difficulty increasingly moves from pointwise identity toward relations and quantifiers.

VII. Two Isolated Peaks Enter One Mountain Range: The Core Results of This Book

To say only that this book studies Goldbach and Twin would underestimate it.

To say only that it proposes a “unified framework” would underestimate it as well.

The first description sees only the peaks; the second sees only the map.

What this book seeks to accomplish is: **let the map explain the peaks, and let the peaks test the map.**

Across its seven parts, the book first establishes a common object independent of the names Goldbach or Twin: the fixed arithmetic identity spectrum, multiplicatively irreducible atoms, a continuously expanding atomic relation matrix, finite moving horizons, relation-faithful sieves, relation quantifiers, and finite certificates. Goldbach and Twin then enter the proof chain as the two most fundamental classical readouts of this matrix.

Within the theory developed here, Goldbach is no longer treated as an isolated “additive miracle.” It is identified as a **fixed-sum universal fiber** inside the same two-atom relation field:

$$\boxed{\text{Goldbach} = \text{fixed sum} + \text{nonemptiness of every target fiber.}}$$

Its proof unfolds along

$$\boxed{\text{atomic generation} \to \text{full-row birth of relations} \to \text{fixed-sum slicing} \to \text{moving horizon} \to \text{terminal accounting} \to \text{universal}}$$

Each target enters its own finite terminal world; the identity, death, filling, and final readout of relation slots acquire determinate expression within a finite history. Goldbach therefore becomes the representative case of what this book calls a **terminal-saturation relation sieve**.

Twin is likewise no longer treated as an entirely separate prime-number world. It uses the same atom set, the same birth history of relations, and the same sum–difference coordinates as Goldbach, but reads the other axis:

$$\boxed{\text{Twin} = \text{unit fixed gap} + \text{infinite recurrence.}}$$

Its proof does not depend on a common terminal horizon. Instead, it tracks the generation, elimination, permanent preservation, and cross-horizon accumulation of genuine two-atom relations through consecutive square moving horizons, turning "infinity" into the unbounded extension of finite relation certificates. Twin therefore becomes the representative case of a **persistent non-extinction relation sieve**.

Thus two classical propositions that had been separately named and separately advanced for centuries enter one and the same relation plane in this book:

$$\boxed{\begin{aligned}\text{Goldbach:}&\quad \sigma = \text{fixed}, \quad \forall\sigma;\\ \text{Twin:}&\quad \delta = 1, \quad \exists^{\infty}.\end{aligned}}$$

Their difference no longer appears as "requiring two unrelated foundational theories," but as **different slices and different quantifiers on the same matrix**.

This is also the most central statement of the book's achievements:

> **The proofs of Goldbach and Twin are the two most visible results of Unified Prime Relations Theory;**
> **the unified relation matrix is what makes it possible to understand those two results together.**

And the book does not stop with these two propositions.

Part VI further compiles fixed sums, fixed differences, weighted reflections, prime tuples, arithmetic-progression chains, windows, neighboring-prime relations, affine relations, polynomial orbits, sparse exponential orbits, complete prime-factor sets, and global spectral relations into the common form "relation orbit—target identity—observation region—global quantifier." In this way, a family of classical theorems and propositions receives unified formal proofs, equivalent reconstructions, and certificate systems; problems formerly scattered across different chapters and traditions are returned to one common object language.

Therefore, the achievements of this book cannot be summarized merely as "proving two conjectures."

A more accurate formulation is:

> **This book establishes a Unified Prime Relations Theory, with Goldbach, Twin,**
> **and a family of classical prime propositions as its central proofs and readouts.**

Every new conclusion in the book rests ultimately on the definitions, theorem chains, finite certificates, and executable verification presented in the main text. The theory does not ask the reader to accept an announcement; it requires every conclusion to be capable of being unfolded again along its own relation history.

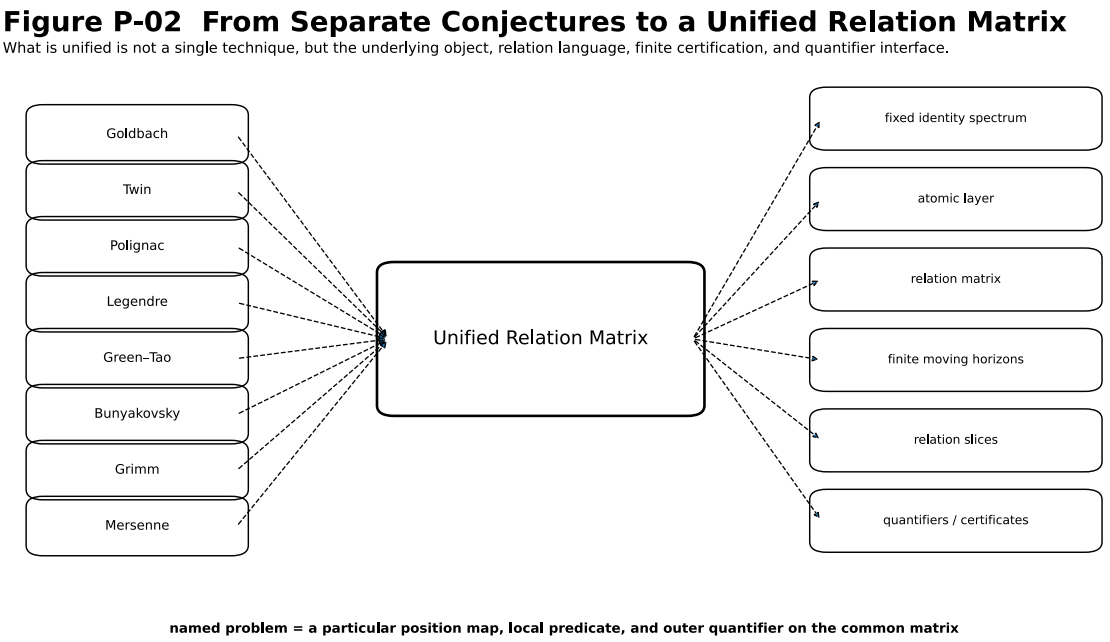


Figure P-02. From Separate Conjectures to a Unified Relation Matrix. Named problems such as Goldbach, Twin, Polignac, Legendre, Green–Tao, Bunyakovsky, Grimm, and Mersenne are returned to a common structure of fixed identity, atomic layers, a relation matrix, finite moving horizons, relation slices, and quantifier/certificate interfaces. What is unified is the object and language layer, not a compulsory single method.

VIII. The Real Turning Point: From "One Conjecture, One Language" Back to a Common Matrix

Over the past several centuries, prime-number research has developed in a completely natural organizational order:

first specify the prime set → state a famous problem → seek tools suited to that problem.

This approach has produced magnificent achievements.

But as the number of problems grew and the methods deepened, a new organizational problem emerged. Goldbach had the language of Goldbach; Twin had the language of Twin; Legendre had the language of windows; Bunyakovsky had the language of polynomials; Green–Tao had the language of arithmetic progressions; Grimm and Erdős–Woods entered the language of complete factor sets. The names of problems gradually came to precede their common underlying object.

This book changes first of all the order of organization.

Its starting point becomes:

First come the fixed arithmetic identities of integers;
then relations read those identities;
only after that do propositions impose quantifiers on the relations.

The entire book therefore obeys three basic layers:

L_0: fixed identity,	L_1: relational readout,	L_2: global quantifier.

They can be compressed into one sentence:

The matrix provides identity; relations choose positions; propositions specify quantifiers.

From here onward, prime-number problems need not begin from their historical names.

They can first return to the common matrix, then reappear in the form of their genuine relational structure.

IX. Part I: The Fixed Arithmetic Identity Spectrum and the Atomic Relation Matrix

Part I establishes the lowest-level object of the entire book.

First, every integer carries a fixed arithmetic identity: whether it is prime, what its least prime factor is, what its multiplicative depth $\Omega(n)$ is, and which distinct prime factors it possesses. This gives the fixed arithmetic identity spectrum

$$\boldsymbol{I}(n).$$

The spectrum answers the most basic question:

> *What is each integer?*

The book then writes positive odd integers in the shared coordinate system

$$\Phi(a) = 2a + 1,$$

and defines

$$a \star b = a + b + 2ab, \qquad a \boxplus b = a + b + 1.$$

These operations carry odd multiplication and addition of two odd integers exactly:

$$\Phi(a \star b) = \Phi(a)\Phi(b),$$

$$\Phi(a) + \Phi(b) = 2(a \boxplus b).$$

Within this same coordinate system, odd primes are identified as irreducible atoms of $(\mathbb{N}_0,\star)$. Each atom emits its own multiplicative ray; every non-atomic position is covered by atomic rays; the least escape point beyond the already generated rays becomes the next atom.

More importantly, the birth of a new atom does not merely add one point.

It immediately forms an entire new layer of two-atom relations with all historical atoms:

$$\Gamma_k = \Gamma_{k-1} \sqcup \Lambda_k.$$

The prime sequence is thereby lifted into a continually expanding relation matrix in which history is never lost.

Then, through the lossless sum–difference coordinates

$$\Psi(r) = (\sigma, \delta),$$

the same pair of atoms simultaneously acquires a sum readout and a difference readout.

The common foundation of Goldbach and Twin becomes directly visible.

X. Part II: Finite Moving Horizons—the Finite Entry Point to Infinite Relations

Once the unified matrix is established, a key question immediately arises: the matrix grows without bound, so how can a concrete relation receive terminal certification from finite information?

Part II establishes the **finite moving horizon**.

For any finite upper bound X, whether an odd integer in the window is prime can be decided using only historical atomic layers not exceeding $\sqrt{X}$. The classical square-root criterion is thereby elevated into the book's unified principle of terminal certification:

finite window + atomic layers up to $\sqrt{X} \Rightarrow$ all identities in the window are terminally determined.

Candidate relations are therefore no longer left indefinitely unresolved.

A candidate endpoint is either genuinely eliminated at some finite layer by a multiplicative factor, or it completes certification inside its own finite horizon and is promoted to a permanent relation. Once a confirmed relation enters the matrix, future atoms cannot change it.

This part also establishes relation fidelity, genuine death, the first killer, relation histories, persistent embeddings, and a unified closure mechanism that carries finite horizons into universal and infinite quantifiers.

A finite moving horizon is not a crude truncation of infinity.

It is the finite entry point through which an infinite relation becomes subject to rigorous mathematical verification.

XI. Part III: Relation Slices, Derived Geometry, and Quantifier Structures

Part III expands the two-atom matrix into relation geometry.

Let

$$m = a + b + 1, \qquad h = b - a.$$

In ordinary prime coordinates, the two endpoints are

$$p = m - h, \qquad q = m + h.$$

Fixed sums, fixed differences, fixed larger atoms, and fixed products therefore become different slices, directions, and curves in one relation plane.

Goldbach and Twin first appear side by side here in an extremely compact form:

Goldbach is a fixed-sum slice; Twin is a unit-fixed-gap slice.

Part III further extends the language to finite patterns, affine relations, arithmetic-progression chains, prime constellations, windows, and general finite-order relations, while carefully distinguishing different global quantifiers: infinite recurrence, parameterwise universality, window universality, arbitrary length, infinitely many parameters, indexwise universality, and identity-block separation.

"Infinity" is no longer treated as one vague word.

Each kind of infinity must be matched with its own finite certificate.

XII. Part IV: Finite-Control Geometry and the Terminal-Saturation Structure of Goldbach

Part IV enters the finite-control layer of relation sieves.

Local obstruction, CRT survival trees, first local elimination, genuine death, relation first-killers, relation alphabets, and relation-faithful sieves are unified within one finite-control geometry.

In the fixed-sum direction, every target possesses its own natural terminal segment. Once relation slots enter the terminal region, endpoint identities, death responsibility, filling mechanisms, and final readouts are completely determined by finite history.

This part returns additive relations and multiplicative occupancy to the same natural-number matrix: multiplication does not "create another world," and addition does not create prime identity; a fixed sum merely repairs already fixed identities into a prescribed pairing relation.

From this arise the core structures of natural terminal freezing, partition of terminal segments, exhaustion of relation slots, slotwise exact filling, conservation of the global relation-slot count, decomposition of dual multiplicative responsibility, and the terminal readout of the universal Goldbach relation.

Goldbach therefore no longer appears as an isolated proposition detached from the matrix, but as the paradigmatic **terminal-saturation type** in a unified relation sieve.

XIII. Part V: The Exact Architecture of Infinite Relations and the Persistent Non-Extinction Structure of Twin

A fixed-gap relation has no common terminal horizon.

Each concrete candidate pair can be certified in a finite square horizon, but the unit-gap relation itself can keep moving to larger positions. Part V therefore studies a second basic type: the **persistent non-extinction relation sieve**.

Consecutive prime squares form natural moving boundaries; inside each square horizon, relation supply, genuine relations, genuine deaths, and finite absorption resources can all be recorded item by item. From this are built unique square-horizon assignment, square-seam closure, one-layer purification, resource half-life, resource lifetime, absorption reuse degree, net relation potential, and a cross-horizon cumulative ledger.

Twin becomes the basic instance of the smallest fixed pattern $H = \{0,2\}$.

"Infinite twin primes" is no longer described as an infinite object that cannot be grasped as a whole, but is reduced to

> how genuine relation certificates inside finite horizons
> accumulate without bound across horizons.

The difference between Goldbach and Twin can therefore be expressed clearly inside one and the same system:

> Goldbach: terminal saturation;
> Twin: persistent non-extinction.

XIV. Part VI: Unified Compilation of Classical Propositions and the Boundaries of Methods

Part VI systematically unfolds the carrying capacity of the unified matrix.

Goldbach, Lemoine/Lévy, weak Goldbach, Chen-type relations, Twin, Polignac, Sophie Germain/safe primes, Hardy–Littlewood prime k-tuples, Green–Tao, Dirichlet, the PNT and the PNT in arithmetic progressions, Bertrand, Legendre, Oppermann, Brocard, Andrica, Firoozbakht, $n^2 + 1$, Bunyakovsky, Schinzel H, Mersenne, Fermat, Waring–Goldbach, Grimm, Erdős–Woods, together with spectral interfaces to RH/GRH, are each sent into a unified relation normal form.

The core compilation format is

$$\boxed{\text{relation orbit} + \text{target identity depth} + \text{observation region} + \text{global quantifier.}}$$

This gives classical propositions a common "type system."

At the same time, the Sieve of Eratosthenes, Brun sieve, Selberg sieve, combinatorial sieve, linear sieve, large sieve, Bombieri–Vinogradov, the Hardy–Littlewood circle method, Chen's method, GPY, Maynard–Tao, Green–Tao, analytic L-functions, and finite primality certificates such as Pocklington, APR, and ECPP are placed on one common map of methodological capabilities.

Here the book states its boundary of unification explicitly:

> **Unify the object layer; do not force all concrete tools into one method.**

Different methods preserve different levels of information. The responsibility of a unified theory is not to turn them into one method, but to tell us precisely which layer of the same matrix each method acts upon.

XV. Part VII: RenSpectrum—Bringing the Theory into an Executable World

A theory that is to become a common basis for research cannot remain only a narrative.

Part VII builds RenSpectrum—the Unified Prime Relations Workbench.

It does not merely recreate a table of primes. Rather, it compiles the fixed identity spectrum, relation normal forms, finite horizons, quantifier types, relation certificates, seam structures, supply–absorption modules, and validation levels into executable objects.

Goldbach, Twin, prime tuples, fixed Green–Tao slices, Legendre, Oppermann, Brocard, Mersenne, $n^2 + 1$, Bunyakovsky, Schinzel H, Grimm, Erdős–Woods, and user-defined relations can all enter the same relation language.

The book therefore ends with a continuous three-layer structure:

$$\boxed{\text{mathematical theory} \rightarrow \text{formal relation language} \rightarrow \text{executable verification.}}$$

A proof need no longer leave behind only its final answer; it can leave behind the objects, rules, births, deaths, terminal horizons, certificates, and execution history as well.

This is also the first step from a unified theory toward an open research platform.

Figure P-03 Architecture of the Seven Parts

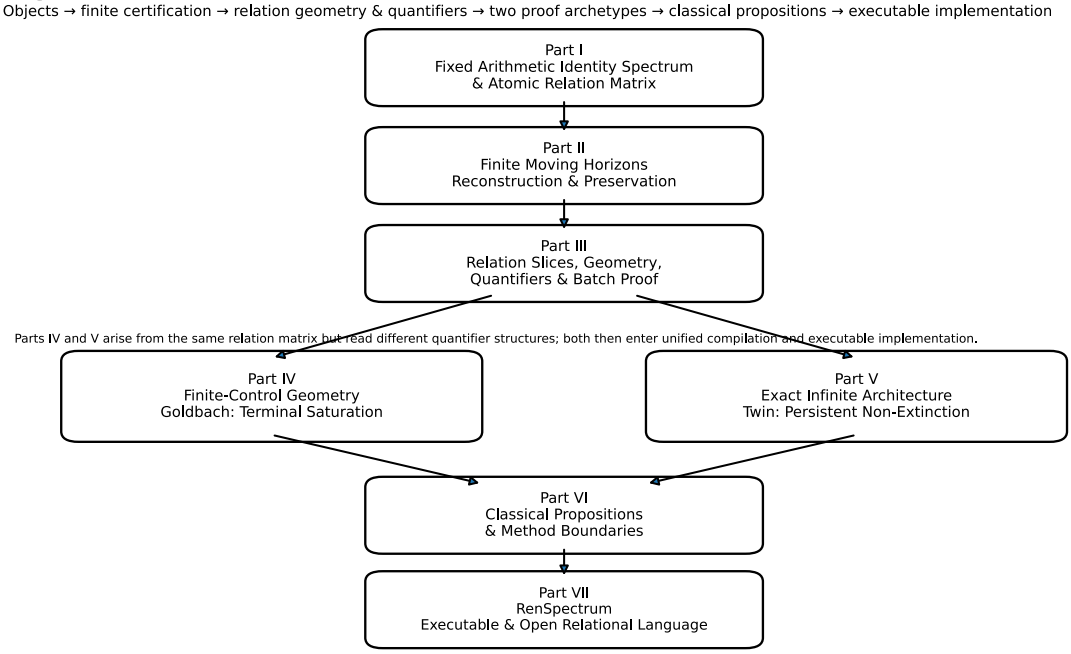


Figure P-03. Architecture of the Seven Parts. The seven parts are successive layers of one object, from fixed identity and finite certification through relation geometry and quantifiers to the two proof archetypes, unified compilation of classical propositions, and the executable RenSpectrum implementation.

XVI. From Two Classical Propositions to a Theory of Relations

This book places Goldbach and Twin prominently in its title not merely because they are famous. They happen to represent two of the most fundamental global requirements in prime relations:

$$\forall \qquad \text{and} \qquad \exists^{\infty}.$$

Goldbach requires that no target be absent.

Twin requires that the same relation never cease to occur.

A theory capable of treating only one of these may still be merely a local method. Unified Prime Relations Theory acquires its own subject matter only when the same matrix can carry both quantifiers and each can be supplied with its matching finite proof and global closure.

But this is still not the endpoint.

Once fixed identity, relation orbit, moving horizon, and quantifier certificate have been separated into layers, a new prime-number problem entering the theory need not rebuild the underlying world. It need only answer:

- Which positions does it read?
- What identities does it require?
- How is its relation preserved?
- In which finite horizon is it certified?
- Which global quantifier does it belong to?

- What finite certificate matches that quantifier?

Then “a conjecture” is no longer the only possible unit of research.

A more foundational unit becomes

the relation type and its certificate.

This is the theoretical shift the book seeks to leave behind.

XVII. Before Reading

Four sentences are useful to keep in mind throughout the book.

First:

> **Fixed identity is not the same as relational fact.**

Second:

> **Whenever a relation is compressed, one must prove that no essential witness has been lost.**

Third:

> **A finite moving horizon is not an approximation to an infinite relation, but its rigorous finite entry point.**

Fourth:

> **Unify the foundation without erasing differences; unify the objects without excluding tools.**

Entering through Goldbach, one can follow fixed sums and see how the atomic relation matrix enters terminal segments, relation slots, genuine death, exact filling, and universal readout.

Entering through Twin, one can follow the unit fixed gap and see how relations cross consecutive square horizons, passing from finite certificates to infinite recurrence.

Entering from analytic number theory, one can go directly to the method boundaries in Part VI and see what information is preserved, at the unified object layer, by the circle method, sieve methods, the large sieve, Bombieri–Vinogradov, GPY, Maynard–Tao, Green–Tao, and L-functions.

Entering from computation, one can read the first six parts backward from RenSpectrum in Part VII: every executable field corresponds to a mathematical object already defined in the theory.

And if one begins with a new problem that has never yet been named, this book proposes a different opening move:

> Do not first ask which famous conjecture it resembles; first ask what relation it reads from the fixed spectrum.

For thousands of years, people have kept asking where the primes are.

This book tries to move one step further:

> **The primes have always been there; what must be seen anew is how they form relations.**

Terminology Glossary — Unified Prime Relations Theory

This glossary fixes the principal English terminology used throughout the complete translation.

Chinese	Standard English
统一素数关系理论	Unified Prime Relations Theory
固定算术身份谱	fixed arithmetic identity spectrum
算术身份	arithmetic identity
原子	atom
原子关系母体	atomic relation matrix
生成式原子关系母体	generative atomic relation matrix
共享奇数坐标	shared odd-number coordinate system
乘法压缩	multiplicative compression
不可约释放	irreducible release
原子递归生成	recursive generation of atoms
原子射线	atomic ray
最小逃逸点	least escape point
关系整行出生	full-row birth of relations
关系出生层	relation birth layer
持久继承	persistent inheritance
持久关系	persistent relation
关系母体	relation matrix
母—子读出	matrix-to-subrelation readout
关系切片	relation slice
和—差坐标	sum–difference coordinates
无损关系坐标	lossless relation coordinates
关系还原	relation reconstruction
无损关系描述子	lossless relation descriptor
目标充分描述子	target-sufficient descriptor
关系保真状态	relation-faithful state
有限移动视界	finite moving horizon
平方视界	square horizon
终端认证	terminal certification
零误差终端认证	zero-error terminal certification
候选关系壳	candidate relation shell
永久关系骨架	permanent relation skeleton
真实死亡	genuine death
死亡层	death layer
关系死亡层	relation death layer
首次局部淘汰	first local elimination
首杀因子 / first-killer	first killer
关系履历	relation history
关系几何	relation geometry
导出几何	derived geometry
量词结构	quantifier structure
量词闭合	quantifier closure
全称闭合	universal closure
全称非空	universal nonemptiness
逐参数全称	parameterwise universality
无限回归	infinite recurrence
累计计数无界	unbounded cumulative count

Chinese	Standard English
有限证书	finite certificate
批量论证	batch proof
有限控制几何	finite-control geometry
局部阻塞	local obstruction
关系筛	relation sieve
终端饱和型关系筛	terminal-saturation relation sieve
持续非灭绝型关系筛	persistent non-extinction relation sieve
乘法占位	multiplicative occupancy
原子余项	atomic remainder
关系槽	relation slot
候选槽	candidate slot
终端段	terminal segment
自然终端段	natural terminal segment
关系完备性	relation completeness
供给—吸收	supply–absorption
关系供给	relation supply
吸收能力	absorption capacity
累计净势	cumulative net potential
资源复用	resource reuse
资源半衰	resource half-life
有限寿命	finite lifetime
无限关系	infinite relation
精准架构	exact architecture
关系标准形	relation normal form
形式证明	formal proof
方法边界	method boundary
身份深度	identity depth
素数深度	prime depth
有限模板	finite pattern
仿射关系	affine relation
多项式轨道	polynomial orbit
窗口占据	window occupancy
固定和值	fixed sum
固定差	fixed gap / fixed difference (context-dependent)
Goldbach	Goldbach
Twin / 孪生素数	Twin primes / twin-prime relation
RenSpectrum	RenSpectrum
可执行实现	executable implementation
开放关系语言	open relational language
机器验证	machine verification
独立复核	independent verification
证明链	proof chain
经典命题	classical proposition
经典素数命题	classical prime proposition
底层对象	underlying object
统一读出	unified readout
不同读出	distinct readouts

Part I — The Fixed Arithmetic Identity Spectrum and the Atomic Relation Matrix

— From Integer Identity and the Shared Odd-Number Coordinate System to Full-Row Birth of Relations and Unified Slicing

Purpose of This Part

The first object of prime relations theory is neither a conjecture nor a particular sieve method, but the arithmetic identity already possessed by each integer itself.

On top of this fixed identity, relations involving sums, differences, affine maps, arithmetic progressions, windows, polynomials, consecutive primes, factor sets, and so on are merely different modes of readout.

Accordingly, the first principle of this book is:

The matrix provides identity; relations choose positions; propositions specify quantifiers.

This principle weakens no classical problem. On the contrary, it returns formerly scattered prime-number problems to one fixed spectrum, giving different problems a common, comparable, and batch-processable underlying language.

Introduction — From "Many Prime Problems" Back to "One Fixed Spectrum"

Prime-number problems have a distinctive historical form.

Goldbach concerns a fixed sum of two primes; Twin concerns a fixed difference of two primes; Polignac adds to a fixed difference the requirement that no other prime lie between the endpoints; Sophie Germain concerns

$$p, \qquad 2p+1;$$

Green–Tao concerns arbitrarily long arithmetic progressions of primes; Legendre, Oppermann, and Brocard concern occupancy of primes in different windows; Andrica and Firoozbakht concern numerical relations between consecutive primes; Bunyakovsky and Schinzel H concern prime values along integer maps or polynomial families; Grimm and Erdős–Woods directly read the complete prime-factor identity of integers.

On the surface, these problems belong to different chapters, and even to different technical traditions.

But they share one fact that cannot be bypassed:

Every relation ultimately reads arithmetic identities already possessed by the same integers.

For example, in Goldbach, p does not become prime because it enters the relation

$$p + (N - p) = N.$$

Likewise, in Twin, $p + 2$ does not acquire prime identity merely because it differs from p by 2.

The Legendre window

$$(n^2, (n+1)^2)$$

does not generate the primes inside the window; it only specifies where to read.

Similarly, if a polynomial

$$f(n)$$

takes a prime value at some input, the prime identity belongs to the integer $f(n)$ itself, not to the name "polynomial conjecture."

A more natural order of organization is therefore:

first fix integer identities
↓
then specify readout positions
↓
then combine local relations
↓
finally impose universal, existential, or infinitary quantifiers.

This is the starting point of the book.

A second, equally important underlying structure is obtained by transporting the positive odd integers to the shared coordinate system

$$\Phi(a) = 2a + 1,$$

so that multiplication and the addition of two odd integers are expressed on the same carrier, while odd primes are identified as irreducible atoms of the multiplicative structure. From this one obtains an entire generative structure: atomic rays, least escape points, full-row birth of relations, persistent inheritance of relations, and lossless sum–difference coordinates.

This coordinate structure exists in parallel with the fixed identity spectrum.

Their roles can be distinguished precisely as

fixed integer identity spectrum and atomic generative coordinates.

They are not competing theories.

The former answers:

> *What is each integer?*

The latter answers:

> In the shared coordinate system for odd multiplication, how do these identities emerge from irreducible structure, and how do they naturally generate a relation matrix?

This part unifies the two within one foundational structure.

Abstract

This part establishes the lowest-level objects of Unified Prime Relations Theory: the **fixed arithmetic identity spectrum** and the **atomic relation matrix** built upon it.

First define the integer identity map

$$\mathfrak{I}: \mathbb{Z} \to \mathcal{A},$$

where

$$\mathfrak{I}(n) = \begin{cases} A, & n \in \mathbb{P}, \\ \mathcal{M}_{P^-(n)}, & n > 1 \text{ is composite}, \\ \partial, & n \leq 1. \end{cases}$$

Here A denotes atomic prime identity, $\mathcal{M}_r$ denotes the multiplicative identity whose least prime factor is r, and ∂ denotes the boundary domain containing the nonpositive integers and 1. From the same fixed identity one can also read

$$\alpha(n) = \mathbf{1}_{\mathbb{P}}(n), \qquad P^{-}(n), \qquad \Omega(n), \qquad \mathrm{PF}(n),$$

where $\mathrm{PF}(n)$ is the set of distinct prime factors of n.

We then return to the shared coordinate system for positive odd integers,

$$\Phi(a) = 2a + 1, \qquad a \in \mathbb{N}_0,$$

and define

$$a \star b = a + b + 2ab, \qquad a \boxplus b = a + b + 1.$$

These satisfy, respectively,

$$\Phi(a \star b) = \Phi(a)\Phi(b),$$

and

$$\Phi(a) + \Phi(b) = 2(a \boxplus b).$$

Hence odd primes correspond exactly to irreducible atoms of $(\mathbb{N}_0, \star)$. Each atom a generates the multiplicative ray

$$\mathcal{R}_a = \{a + (2a + 1)b : b \geq 1\};$$

all non-atomic positions are covered exactly by such rays; and the least escape point beyond the first k atomic rays automatically becomes the next atom.

On the relational side, if

$$A_k = \{a_1, \ldots, a_k\},$$

define the complete two-atom relation field

$$\Gamma_k = \big\{\{a_i, a_j\} : 1 \leq i \leq j \leq k\big\}.$$

When the k-th new atom is born, it generates the entire new relation layer

$$\Lambda_k = \{\{a_i, a_k\} : 1 \leq i \leq k\},$$

with the exact disjoint decomposition

$$\Gamma_k = \Gamma_{k-1} \sqcup \Lambda_k.$$

This yields the persistent system

$$\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \cdots, \qquad \Gamma_\infty = \varinjlim_k \Gamma_k.$$

For a two-atom relation

$$r = \{a_i, a_j\}, \qquad i \leq j,$$

define the lossless sum–difference coordinates

$$\Psi(r) = (\sigma(r), \delta(r)) = (a_i + a_j + 1,\ a_j - a_i),$$

with the unique inversion

$$a_i = \frac{\sigma - 1 - \delta}{2}, \qquad a_j = \frac{\sigma - 1 + \delta}{2}.$$

Finally, this part unifies the “fixed spectrum–relation slice” principle with the atomic relation matrix: integer identities belong to the fixed spectrum; relations only specify where to read; classical prime-number problems are different slices of that fixed spectrum. Goldbach, Twin, Green–Tao, Legendre, Mersenne, Grimm, and others no longer require separate underlying objects of their own.

The overall structure established here is

fixed identity → multiplicatively irreducible atoms → recursive generation of atoms → full-row birth of relations → persiste

Keywords: fixed arithmetic identity spectrum; prime atoms; shared odd-number coordinates; relation matrix; multiplicative rays; least escape point; relation birth; persistent inheritance; sum–difference coordinates; relation slices

Chapter 1 — The First Object: The Fixed Arithmetic Identity Spectrum

1.1 The Integer Line Comes Before Any Particular Relation

Let

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}.$$

Every prime relation discussed in this book ultimately lives on this integer line.

Whether a problem is written as

$$p + q = N,$$

or

$$q - p = d,$$

or

$$p,\ p + d,\ p + 2d, \dots, p + (k-1)d,$$

these formulas first of all specify relations among certain integer positions.

Once the positions are fixed, their prime, composite, and factor identities have already been determined by the multiplicative structure of the integers.

Thus identity is defined first, and relation second.

Definition 1.1 — Fixed Arithmetic Identity Spectrum

Let the identity alphabet be

$$\mathcal{A} = \{\partial, A\} \cup \{\mathcal{M}_r : r \in \mathbb{P}\}.$$

Define

$$\boxed{\mathfrak{I}(n) = \begin{cases} A, & n \in \mathbb{P}, \\ \mathcal{M}_{P^-(n)}, & n > 1 \text{ is composite}, \\ \partial, & n \le 1. \end{cases}} \tag{1.1}$$

Here $P^-(n)$ denotes the least prime factor of n.

Call

$$\boxed{(\mathbb{Z}, \mathfrak{I})}$$

the **fixed arithmetic identity spectrum**.

The word “fixed” has a precise meaning:

> For every integer n, $\mathfrak{I}(n)$ does not change when one later chooses Goldbach, Twin, a window, an affine relation, a polynomial, or any other relation.

Figure 1-01 Multi-Field Fixed Arithmetic Identity Spectrum

Each integer position carries fixed arithmetic identity; relations read these fields rather than create them.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Prime indicator α(n)	1	1	0	1	0	1	0	0	0	1	0	1	0	0
Least prime factor P-(n)	2	3	2	5	2	7	2	3	2	11	2	13	2	3
Multiplicative depth Ω(n)	1	1	2	1	2	1	3	2	2	1	3	1	2	2
Prime-factor set PF(n)	{2}	{3}	{2}	{5}	{2,3}	{7}	{2}	{3}	{2,5}	{11}	{2,3}	{13}	{2,7}	{3,5}

Only 2–15 are shown. The full spectrum contains every integer position; different relation problems read the same spectrum.

Figure 1-01. Multi-Field Fixed Arithmetic Identity Spectrum. Each integer position simultaneously carries fixed fields such as primality, least prime factor, multiplicative depth, and its prime-factor set. Later relations select and read these identities; they do not alter them.

1.2 The Primality Indicator Is Only a Projection of the Identity Spectrum

Define

$$\boxed{\alpha(n) = \mathbf{1}_{\{n\in\mathbb{P}\}}.} \tag{1.2}$$

Then

$$\alpha(n) = 1 \Leftrightarrow \Im(n) = A.$$

Thus α is not a second spectrum; it is the binary projection

$$\Im \to \{0,1\}.$$

When a problem asks only whether an integer is prime, α is enough.

When a problem also asks why a composite position dies, which least prime factor is responsible, or how complete prime-factor sets match, richer identity fields must be read.

Definition 1.2 — Least-Prime-Factor Field

For $n > 1$, define

$$\boxed{\ell(n) = \begin{cases} n, & n \in \mathbb{P}, \\ P^{-}(n), & n \text{ is composite.} \end{cases}} \tag{1.3}$$

In an actual relation sieve, if n is composite, then $P^{-}(n)$ gives the unique first-responsibility prime for that position.

Definition 1.3 — Multiplicative Depth

If

$$n = p_1 p_2 \cdots p_r$$

with prime factors counted with multiplicity, define

$$\boxed{\Omega(n) = r.} \tag{1.4}$$

Also define the Liouville parity character

$$\boxed{\lambda(n) = (-1)^{\Omega(n)}.} \tag{1.5}$$

These fields distinguish different multiplicative depths inside relation sieves.

Definition 1.4 — Complete Prime-Factor Set

Define

$$\boxed{\mathrm{PF}(n) = \{p \in \mathbb{P} : p \mid n\}.} \tag{1.6}$$

Multiplicity is not counted here.

For example,

$$\mathrm{PF}(12) = \{2,3\},$$

$$\mathrm{PF}(18) = \{2,3\},$$

whereas

$$\Omega(12) = 3, \qquad \Omega(18) = 3.$$

For problems of Grimm or Erdős–Woods type, reading $\alpha(n)$ is no longer sufficient; one must read $\mathrm{PF}(n)$.

1.3 A Single Spectrum Can Carry Multiple Fields

Thus an integer position n can be understood as carrying the fixed identity record

$$\boxed{\boldsymbol{I}(n) = (\Im(n), \alpha(n), P^{-}(n), \Omega(n), \mathrm{PF}(n), \dots).} \tag{1.7}$$

Different problems call only the fields they need.

This gives the first foundational discipline of the book:

> **The same integer has only one arithmetic identity;**
> **different relations merely read different fields.**

(1.8)

Chapter 2 — Identity and Relation: Two Levels That Cannot Be Interchanged

2.1 Relations Select Positions; They Do Not Produce Positional Identity

Let X be a parameter or anchor space.

A relation first specifies a position map

$$f: X \to \mathbb{Z}.$$

If several positions are involved, write

$$f_1, \dots, f_m.$$

For example, in a fixed-gap relation,

$$f_1(p) = p, \qquad f_2(p) = p + 2.$$

In a fixed-sum relation,

$$f_1(p; N) = p, \qquad f_2(p; N) = N - p.$$

What the relation does is

$$x \mapsto (f_1(x), \dots, f_m(x)).$$

Only afterward are the identity readings taken:

$$\boldsymbol{I}(f_1(x)), \dots, \boldsymbol{I}(f_m(x)).$$

These two steps must remain separate.

Principle 2.1 — Identity-First Principle

For any position map f,

$$\boxed{\boldsymbol{I}(f(x))}$$

is determined by the integer $f(x)$ itself, not by the name of the map f.

Therefore,

A relation operator selects positions; it does not produce positional identity.

(2.1)

2.2 Three Levels

To standardize the language of the whole book, divide the structure into three levels.

L0: Fixed-Spectrum Layer

$$\boxed{(\mathbb{Z}, \boldsymbol{I}).}$$

This layer answers only:

> *What is a given integer?*

L1: Relational Readout Layer

Given position maps, finite logic, comparisons, windows, counting operations, and so on, read the fixed spectrum.

This layer answers only:

> *Where should we look, and how should these local readings be combined?*

L2: Propositional Quantifier Layer

Impose on the local relations of L1 the quantifiers

$$\forall, \qquad \exists, \qquad \exists^{\infty},$$

or equivalent quantifier structures.

This layer answers only:

> *For how many parameters is the local relation required to hold?*

Thus

$$\boxed{L0 \to L1 \to L2.} \tag{2.2}$$

The reverse direction cannot rewrite underlying identity:

$$\boxed{L2 \nrightarrow L0.} \tag{2.3}$$

This is the basic form of every "matrix–subrelation" structure in the book.

Figure 1-02 The Non-Interchangeable L0–L1–L2 Hierarchy

L0 supplies factual objects; L1 organizes relations; L2 imposes proposition-level quantifiers. Higher layers do not rewrite lower ones.

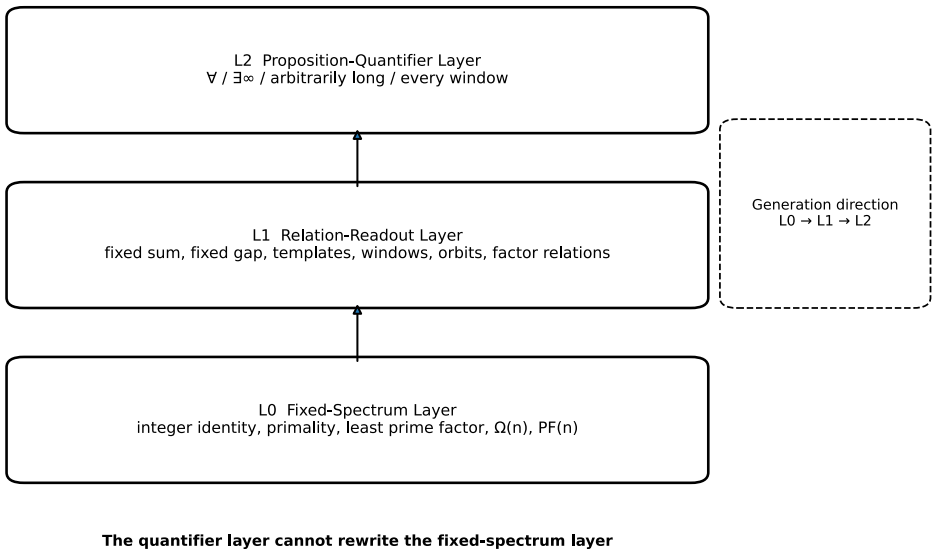


Figure 1-02. The Non-Interchangeable L0–L1–L2 Hierarchy. L0 supplies fixed identities; L1 organizes relations over those identities; L2 imposes proposition-level quantifiers such as

universality, infinite recurrence, or window universality. Construction proceeds upward, but the quantifier layer cannot rewrite the fixed-spectrum layer.

Chapter 3 — Why Atomic Coordinates Are Still Needed

The fixed identity spectrum tells us what every integer is.

But that alone does not answer another important question:

> *What structural meaning does prime identity have inside odd multiplication?*

In the shared odd-number coordinate system, this question has a compact answer.

The coordinate change transports the positive odd integers to natural-number coordinates, placing multiplication and the addition of two odd integers on the same carrier.

This coordinate system is not a second identity spectrum.

It is another exact coordinate description of the same integer world.

Definition 3.1 — Shared Odd-Number Coordinates

Let

$$\mathbb{N}_0 = \{0,1,2,\dots\}.$$

Define

$$\boxed{\Phi(a) = 2a + 1.} \tag{3.1}$$

Since α is reserved throughout for primality on ordinary integers, define the atomic-coordinate indicator

$$\boxed{\hat{\alpha}(a) := \alpha(\Phi(a)) = \mathbf{1}_A(a).} \tag{3.1a}$$

and its complementary indicator

$$\boxed{\hat{\beta}(a) = 1 - \hat{\alpha}(a).} \tag{3.1b}$$

Hence

$$\hat{\alpha}(a) = 1 \Leftrightarrow a \in A \Leftrightarrow 2a + 1 \in \mathbb{P}.$$

From this point onward, $\alpha(n)$ acts only on ordinary integers n, while $\hat{\alpha}(a)$ acts only on atomic coordinates a.

The map

$$\Phi\colon \mathbb{N}_0 \to \{1,3,5,\dots\}$$

is a bijection.

Thus odd positions on the integer axis correspond one-to-one with atomic coordinates a.

3.1 The Integer Spectrum and the Atomic Spectrum Are Two Views of the Same Object

Define atomic-coordinate identity by

$$\boxed{\mathfrak{I}_A(a) = \mathfrak{I}(\Phi(a)).} \tag{3.2}$$

Thus

$$\mathfrak{I}_A = \mathfrak{I} \circ \Phi.$$

This shows that

integer coordinates and odd atomic coordinates

are not two independent data sets.

They are the same fixed spectrum expressed in two coordinate systems.

Figure 1-03 Shared Odd-Number Coordinates Φ(a)=2a+1

Atomic natural-number coordinates are in bijection with the positive odd integers; this is not a second spectrum but another coordinate expression of the same fixed spectrum.

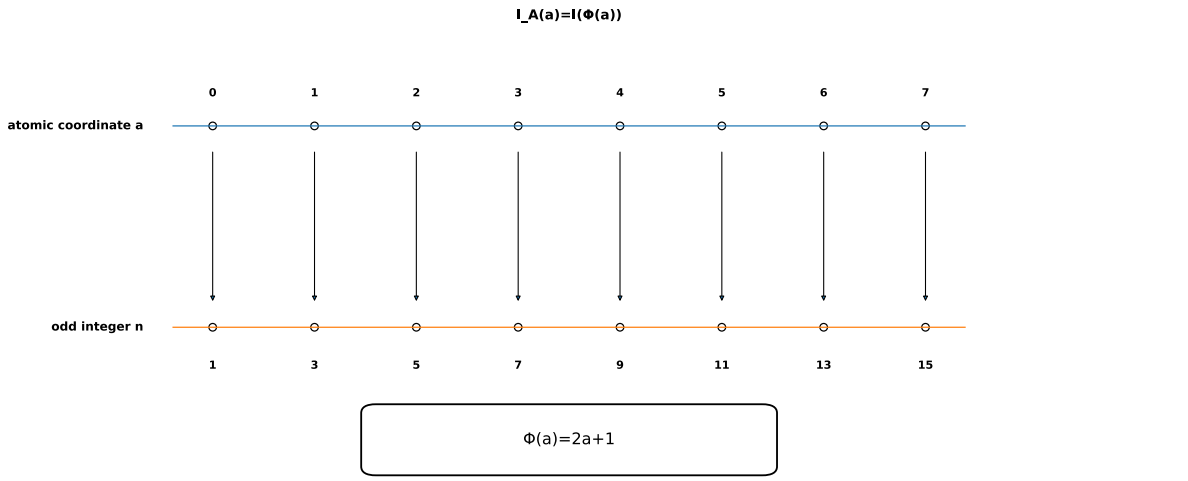


Figure 1-03. Shared Odd-Number Coordinates Φ(a)=2a+1. The map $\Phi(a) = 2a + 1$ is a bijection from $\mathbb{N}_0$ to the positive odd integers. Since $\Im_A(a) = \Im(\Phi(a))$, the integer spectrum and the odd atomic spectrum are two coordinate views of the same fixed spectrum.

Chapter 4 — Two Basic Operations on the Shared Coordinates

Definition 4.1 — Multiplicative Coordinate Operation

For $a, b \in \mathbb{N}_0$, define

$$\boxed{a \star b = a + b + 2ab.} \tag{4.1}$$

Definition 4.2 — Additive Relation Operation

Define

$$\boxed{a \boxplus b = a + b + 1.} \tag{4.2}$$

These two operations place multiplication of positive odd integers and the additive relation between two positive odd integers in the same natural-number coordinate system.

Chapter 5 — Multiplicative Compression: Exact Transport of Odd Multiplication

Theorem 5.1 — Multiplicative Coordinate Isomorphism

For all $a, b \in \mathbb{N}_0$,

$$\boxed{\Phi(a \star b) = \Phi(a)\Phi(b).} \tag{5.1}$$

Proof

By definition,

$$\begin{aligned}\Phi(a \star b) &= 2(a + b + 2ab) + 1 \\ &= 4ab + 2a + 2b + 1 \\ &= (2a + 1)(2b + 1) \\ &= \Phi(a)\Phi(b).\end{aligned}$$

This proves the theorem.

Corollary 5.2 — $(\mathbb{N}_0, \star)$ Is a Commutative Monoid

Its identity element is 0, and

$$\boxed{(\mathbb{N}_0, \star) \cong (2\mathbb{N}_0 + 1, \times).} \tag{5.2}$$

Proof

Commutativity follows from

$$a + b + 2ab = b + a + 2ba.$$

Also,

$$a \star 0 = a.$$

Associativity follows from the bijectivity of Φ and associativity of odd multiplication:

$$\Phi((a \star b) \star c) = \Phi(a)\Phi(b)\Phi(c) = \Phi(a \star (b \star c)).$$

Hence

$$(a \star b) \star c = a \star (b \star c).$$

This proves the corollary.

The operation $\star$ is not an approximate model; it is an exact coordinate transport of odd multiplication.

Chapter 6 — Additive Expansion: Expressing Two-Odd-Integer Relations in the Same Coordinates

Theorem 6.1 — Odd-Integer Addition Formula in Atomic Coordinates

For all $a, b \in \mathbb{N}_0$,

$$\boxed{\Phi(a) + \Phi(b) = 2(a \boxplus b).} \tag{6.1}$$

Proof

$$\begin{aligned}\Phi(a) + \Phi(b) &= (2a + 1) + (2b + 1) \\ &= 2(a + b + 1) \\ &= 2(a \boxplus b).\end{aligned}$$

This proves the theorem.

Thus the same a-coordinate system simultaneously carries

$$\boxed{\star : \text{odd multiplication}, \qquad \boxplus : \text{the additive relation of two odd integers.}} \tag{6.2}$$

This is the precise meaning of "shared coordinates."

Figure 1-04 Two Basic Operations on the Shared Coordinates

The same natural-number coordinate system exactly carries both odd multiplication and two-odd-number addition relations.

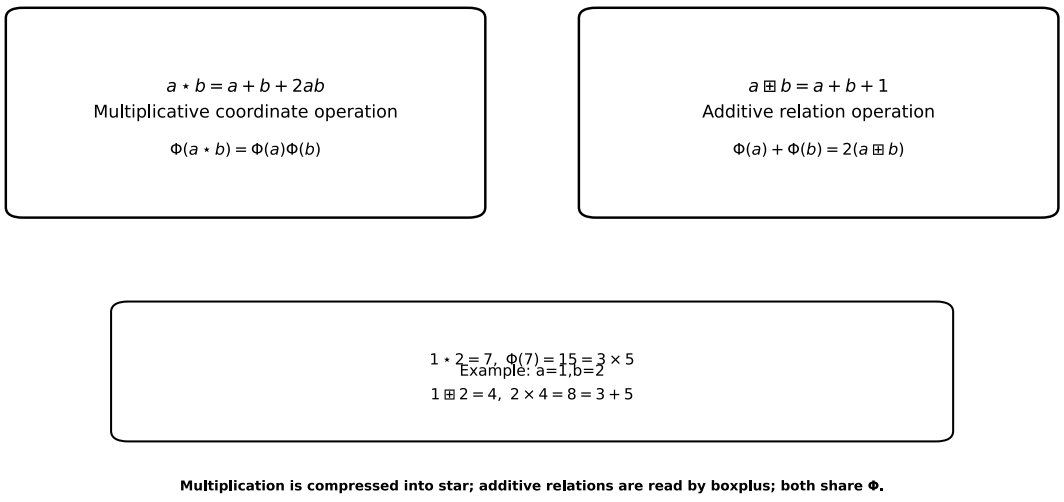


Figure 1-04. Two Basic Operations on the Shared Coordinates. On the same a-coordinates, $\star$ exactly transports odd multiplication via $\Phi(a \star b) = \Phi(a)\Phi(b)$, while $\boxplus$ exactly expresses the sum of two odd integers via $\Phi(a) + \Phi(b) = 2(a \boxplus b)$. Multiplicative generation and additive relations therefore share one carrier.

Chapter 7 — Irreducible Atoms: Odd Primes Emerge from Multiplication Itself

Odd primes are not inserted into $(\mathbb{N}_0,\star)$ as an external list; they are recovered directly from irreducibility.

Definition 7.1 — $\star$-Atom

An element $a \in \mathbb{N}_{\geq 1}$ is called a $\star$-atom if

$$a = x \star y$$

can occur only when

$$x = 0 \quad \text{or} \quad y = 0.$$

Let the set of all $\star$-atoms be

$$\boxed{\mathcal{A}.} \tag{7.1}$$

Theorem 7.2 — Equivalence Between Atoms and Odd Primes

$$\boxed{a \in \mathcal{A} \Leftrightarrow \Phi(a) = 2a + 1 \text{ is an odd prime.}} \tag{7.2}$$

Proof

If $a \notin \mathcal{A}$, then there exist $x, y \geq 1$ such that

$$a = x \star y.$$

By Theorem 5.1,

$$2a + 1 = (2x + 1)(2y + 1),$$

and both factors exceed 1, so $2a + 1$ is composite.

Conversely, if $2a + 1$ is an odd composite number, write

$$2a + 1 = uv, \qquad u, v > 1,$$

with u, v odd. Let

$$u = 2x + 1, \qquad v = 2y + 1.$$

Then $x, y \geq 1$, and

$$\Phi(a) = \Phi(x)\Phi(y) = \Phi(x \star y).$$

Since Φ is injective,

$$a = x \star y.$$

Thus a is not an atom.

This proves the theorem.

7.1 The Position of the Prime 2

The fixed integer spectrum $\mathfrak{I}$ already includes 2 among all prime atoms:

$$\mathfrak{I}(2) = A.$$

The shared odd-number coordinate system $\Phi(a) = 2a + 1$ treats only the odd-number body, so 2 is not in its image.

Accordingly, this book adopts the division of labor

the integer spectrum carries complete prime identity; odd atomic coordinates carry the generative structure of odd multiplication.

(7.3)

The two serve different roles and do not conflict.

Figure 1-05 Equivalence of ★-Atoms and Odd Primes

Irreducibility in the shared coordinates recovers odd-prime identity exactly.

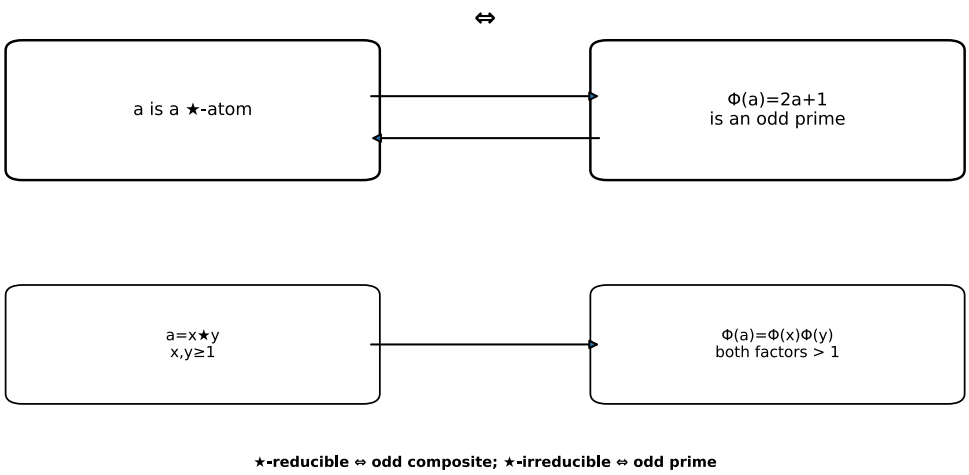


Figure 1-05. Equivalence of ★-Atoms and Odd Primes. By the multiplicative coordinate isomorphism, a admits a nontrivial factorization $x \star y$ if and only if $\Phi(a)$ admits the nontrivial factorization $\Phi(x)\Phi(y)$. Hence $\star$-irreducible atoms correspond exactly to odd primes.

Chapter 8 — Unique Atomic Factorization and Multiplicative History

Theorem 8.1 — Unique ⋆-Atomic Factorization

Every $n \geq 1$ can be written as a finite ⋆-product of ⋆-atoms:

$$\boxed{n = a_1 \star a_2 \star \cdots \star a_r,} \tag{8.1}$$

and the factorization is unique up to permutation.

Proof

By the Fundamental Theorem of Arithmetic,

$$\Phi(n) = p_1 p_2 \cdots p_r,$$

where the p_i are odd primes counted with multiplicity.

Let

$$a_i = \frac{p_i - 1}{2}.$$

By Theorem 7.2,

$$a_i \in \mathcal{A}.$$

Then by Theorem 5.1,

$$\Phi(a_1 \star \cdots \star a_r) = p_1 \cdots p_r = \Phi(n).$$

Since Φ is injective,

$$n = a_1 \star \cdots \star a_r.$$

Uniqueness follows directly from uniqueness of prime factorization in the integers.

This proves the theorem.

8.1 Correspondence Between Atomic Factorization and Identity Fields

If

$$\Phi(n) = p_1 \cdots p_r,$$

then

$$\Omega(\Phi(n)) = r.$$

Thus the factorization length in atomic coordinates agrees exactly with multiplicative depth in the integer spectrum.

Likewise,

$$P^-(\Phi(n))$$

corresponds to the least ⋆-atomic factor.

Hence “multiplicative history” and fixed-spectrum fields correspond exactly:

$$\boxed{\text{atomic factorization} \leftrightarrow \text{integer multiplicative-identity fields.}} \tag{8.2}$$

Chapter 9 — Atomic Rays: Linearization of Multiplicative Generation

Fix an atom $a \in \mathcal{A}$, and let

$$p = \Phi(a) = 2a + 1.$$

Then

$$a \star b = a + b + 2ab = a + (2a + 1)b = a + pb.$$

Theorem 9.1 — Atomic Ray Theorem

All nontrivial multiplicative successors of an atom a form exactly the arithmetic ray

$$\boxed{\mathcal{R}_a = \{a + pb : b = 1,2,3, \dots\}, \qquad p = 2a + 1.} \tag{9.1}$$

Moreover,

$$\boxed{\Phi(a + pb) = p(2b + 1).} \tag{9.2}$$

Proof

This follows directly from

$$a \star b = a + pb$$

and Theorem 5.1.

This proves the theorem.

This gives a very intuitive picture:

> On the multiplicative side, an atom is not an isolated point. In the shared coordinate system, it emits an infinite ray with step size equal to its associated prime.

Chapter 10 — Coverage of Non-Atoms and Recursive Generation of the Next Atom

Theorem 10.1 — Ray-Coverage Theorem

$$\boxed{\mathbb{N}_{\geq 1} \setminus \mathcal{A} = \bigcup_{a \in \mathcal{A}} \mathcal{R}_a.} \tag{10.1}$$

Proof

If

$$n \in \mathcal{R}_a,$$

then

$$n = a \star b$$

for some $b \geq 1$, so n is not an atom.

Conversely, if $n \notin \mathcal{A}$, then $\Phi(n)$ is an odd composite. Choose any odd prime divisor p and write

$$p = 2a + 1, \qquad a \in \mathcal{A}.$$

There exists $b \geq 1$ such that

$$\Phi(n) = p(2b + 1).$$

Hence

$$n = a \star b = a + pb \in \mathcal{R}_a.$$

This proves the theorem.

10.1 Coverage Is Not a Partition

A composite number may have several distinct prime factors, so one coordinate may lie simultaneously on several atomic rays.

For example, a position corresponding to the odd composite

$$3 \cdot 5$$

lies both on the 3-atomic ray and on the 5-atomic ray.

Thus

$$\bigcup_{a \in \mathcal{A}} \mathcal{R}_a$$

is a covering, not a disjoint partition.

This is directly connected with the notion of "genuine death by the least prime factor": when a unique responsible factor is needed, one must read $P^{-}(n)$, not an arbitrary divisor.

Order the atoms as

$$a_1 < a_2 < a_3 < \cdots.$$

Define the union of the first k atomic rays by

$$\boxed{\mathcal{M}_k = \bigcup_{i=1}^{k} \mathcal{R}_{a_i}.} \tag{10.2}$$

Figure 1-06 Atomic Rays and Multiplicative Coverage

$\mathcal{R}_a = \{a + (2a+1)b : b \geq 1\}$; non-atomic positions are covered by atomic rays, and overlaps are allowed.

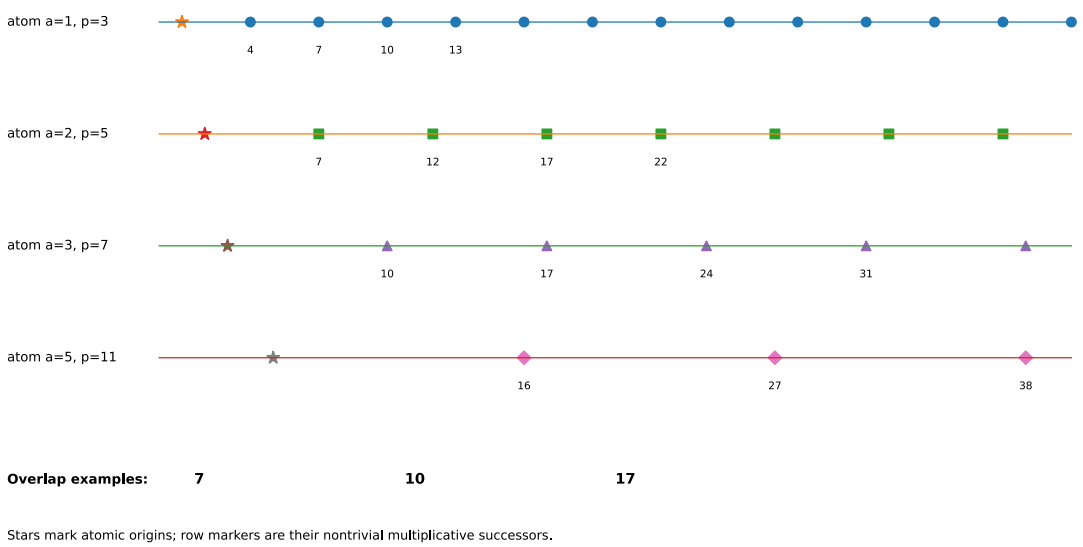


Figure 1-06. Atomic Rays and Multiplicative Coverage. Each atom emits an arithmetic ray with its own prime step. All non-atomic positions are covered by such rays. Different rays may meet at the same position, so this is a covering rather than a disjoint partition.

Theorem 10.2 — Least Escape Point Theorem

$$\boxed{a_{k+1} = \min\{n > a_k : n \notin \mathcal{M}_k\}.} \tag{10.3}$$

Proof

The atom a_{k+1} cannot lie on any previously existing atomic ray; otherwise, by Theorem 10.1, it would be non-atomic.

Conversely, suppose

$$a_k < n < a_{k+1}.$$

There is no new atom in this interval, so n is non-atomic. By Theorem 10.1, n lies on some atomic ray $\mathcal{R}_{a_j}$. Since

$$a_j < n < a_{k+1},$$

and a_{k+1} is the first atom after a_k, we must have

$$a_j \leq a_k.$$

Hence $j \leq k$, and therefore

$$n \in \mathcal{M}_k.$$

This proves the theorem.

Thus atoms can be generated recursively by

existing atoms → emit multiplicative rays → cover all currently reducible positions → the least ∈ (10.4)

This is the rigorous meaning of the book's "generative atomic matrix."

Figure 1-07 The Least Escape Point Generates the Next Atom

$a_{k+1} = \min\{n > a_k : n \notin \mathcal{M}_k\}$. Example: existing atoms 1, 2, 3 yield the next escape point 5.

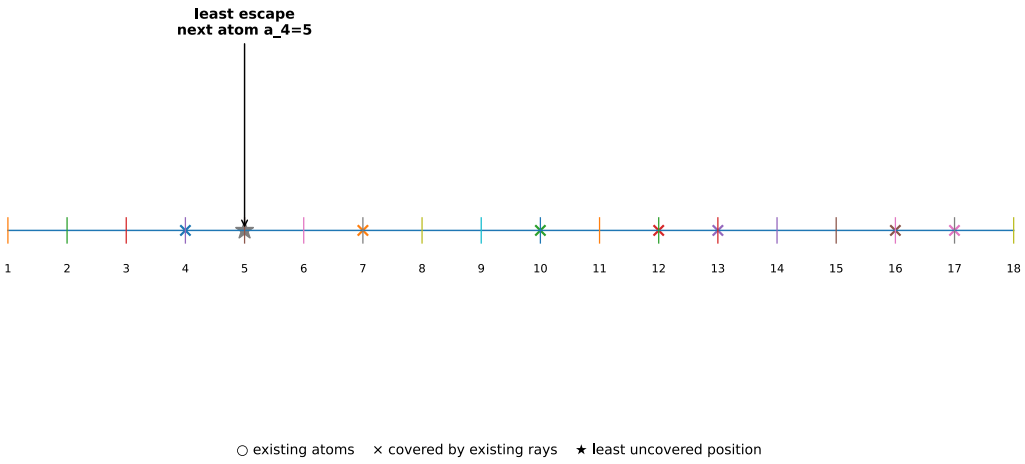


Figure 1-07. The Least Escape Point Generates the Next Atom. After the first k atomic rays have been laid down, the least uncovered position greater than a_k is the next atom a_{k+1}. In the example, atoms 1,2,3 generate the next atom 5.

Chapter 11 — From Atomic Points to Relational Objects

If the only concern were the primes themselves, the preceding construction would already suffice.

But this book studies prime relations.

As soon as a new atom is generated, all of its binary relations with historical atoms are determined as well.

Here one must pay special attention to a hierarchy of information.

If one saves only the sumset

$$A_k \boxplus A_k,$$

then distinct atomic pairs may project to the same sum, and the information about "which two endpoints form this relation" is lost.

The genuine matrix must therefore preserve the **relation instances themselves**.

Definition 11.1 — The Atomic Set at Layer k

Let

$$\boxed{A_k = \{a_1, \dots, a_k\}.} \tag{11.1}$$

Definition 11.2 — The Unordered Two-Atom Relation Field at Layer k

Define

$$\boxed{\Gamma_k = \{\{a_i, a_j\} : 1 \le i \le j \le k\}.} \tag{11.2}$$

We allow $i = j$, so atomic self-pairing is permitted.

Hence

$$\boxed{|\Gamma_k| = \frac{k(k+1)}{2}.} \tag{11.3}$$

The elements here are complete relational objects, not sum labels.

Definition 11.3 — Sum Projection

Define

$$\boxed{\sigma : \Gamma_k \to \mathbb{N}, \qquad \sigma(\{a_i, a_j\}) = a_i + a_j + 1.} \tag{11.4}$$

The corresponding sum-support set is

$$\boxed{S_k = \sigma(\Gamma_k).} \tag{11.5}$$

Therefore,

$$\boxed{\Gamma_k \neq S_k.} \tag{11.6}$$

Γ_k preserves relation instances;

S_k preserves only the support of those relations on the sum axis.

Chapter 12 — The Law of Relation Birth: One New Atom Produces an Entire Row

Definition 12.1 — The k-th Relation-Birth Layer

Define

$$\boxed{\Lambda_k = \{\{a_i, a_k\} : 1 \le i \le k\}.} \tag{12.1}$$

Then

$$\boxed{|\Lambda_k| = k.} \tag{12.2}$$

Theorem 12.2 — Full-Row Birth of Relations

For every $k \ge 2$,

$$\boxed{\Gamma_k = \Gamma_{k-1} \sqcup \Lambda_k.} \tag{12.3}$$

Proof

Every relation in Γ_k either has both endpoints in A_{k-1}, in which case it belongs to Γ_{k-1}, or has at least one endpoint equal to the new atom a_k.

Because the relations are unordered pairs, the latter class is exactly

$$\Lambda_k.$$

The two classes are disjoint, since no relation in Γ_{k-1} contains the endpoint a_k.

This proves the theorem.

Hence

$$\boxed{\Gamma_k = \Lambda_1 \sqcup \Lambda_2 \sqcup \cdots \sqcup \Lambda_k.} \tag{12.4}$$

Definition 12.3 — Relation Birth Time

If

$$r = \{a_i, a_j\}, \qquad i \le j,$$

define

$$\boxed{b(r) = j.} \tag{12.5}$$

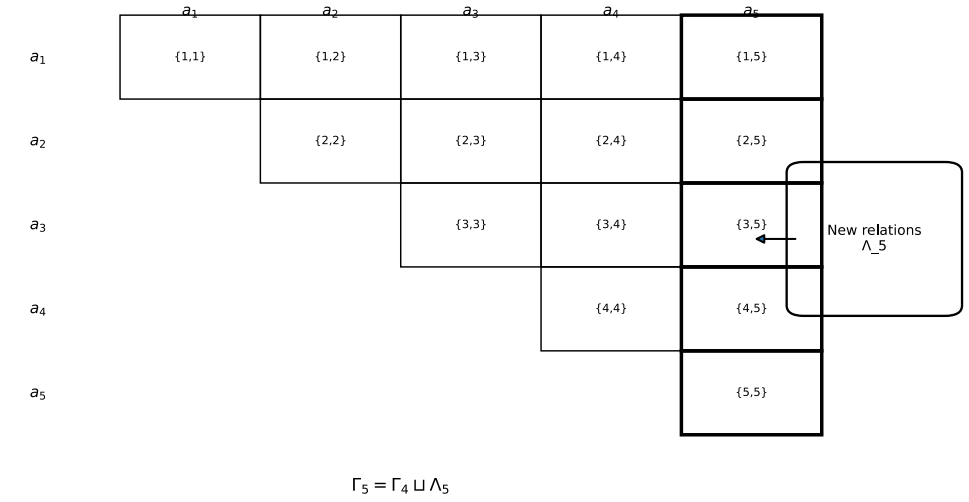
Then

$$r \in \Lambda_{b(r)}.$$

Every relation therefore has a unique birth layer.

Figure 1-08 Birth of an Entire Relation Row

When the new atom a_5 is born, it simultaneously creates relations with all historical atoms a_1,...,a_4; all old relations persist.



$\Gamma_5 = \Gamma_4 \sqcup \Lambda_5$

The thick final column records the new relations from a_5 to all historical atoms; all previous relations are inherited unchanged.

Figure 1-08. Birth of an Entire Relation Row. When a new atom is born, it does not merely add one isolated point. It simultaneously forms an entire new row of relations Λ_k with all historical atoms, while the previous relation set Γ_{k-1} is inherited unchanged.

Chapter 13 — Ray–Relation Duality: Two Expansions Triggered by the Same Atom

When the k-th new atom a_k is born, two completely different updates occur, both triggered by the same atom.

On the multiplicative side:

$$a_k \to \mathcal{R}_{a_k}.$$

On the relational side:

$$a_k \to \Lambda_k.$$

Theorem 13.1 — Ray–Relation Duality Theorem

The birth of the k-th atom has the exact dual update

$$\boxed{\begin{array}{c}\text{multiplicative side: one new infinite multiplicative ray}\\ \Updownarrow\\ \text{the same new atom } a_k\\ \Updownarrow\\ \text{relational side: } k \text{ new two-atom relation instances at the current layer.}\end{array}} \tag{13.1}$$

Thus when the number of atoms increases from $k-1$ to k, the number of relation instances increases by exactly

$$k.$$

Cumulatively,

$$\boxed{1+2+\cdots+k=\frac{k(k+1)}{2}.} \tag{13.2}$$

This is the exact structure of "multiplicative compression–relational expansion":

$$\boxed{\text{one multiplicative escape point} \quad \Rightarrow \quad \text{one entire new relation row.}} \tag{13.3}$$

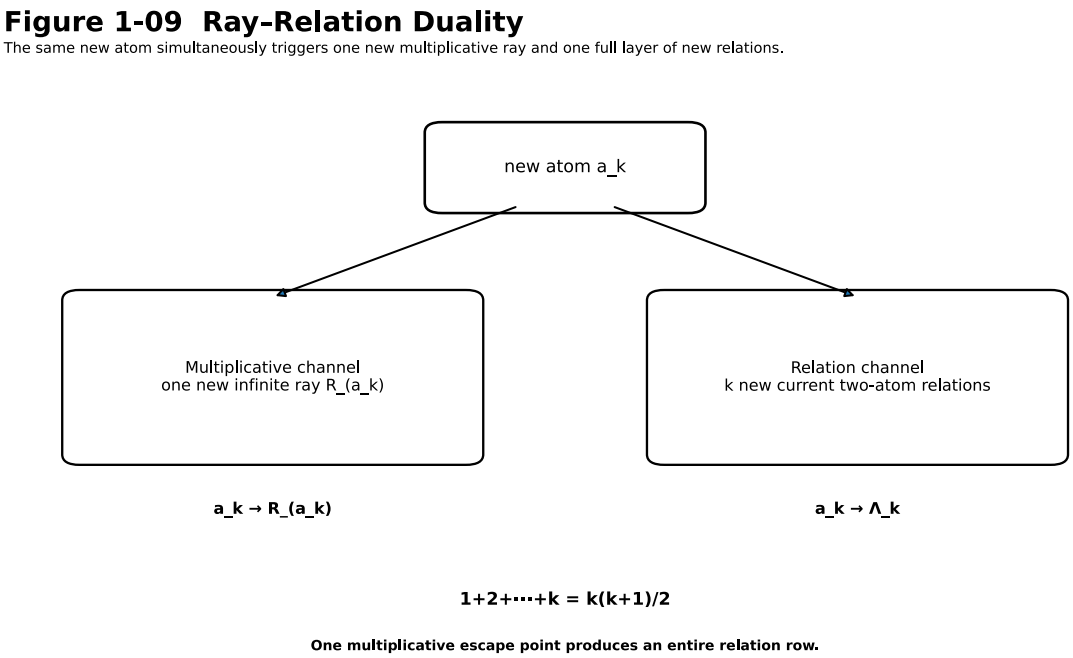


Figure 1-09. Ray–Relation Duality. The same new atom a_k produces one new infinite multiplicative ray $\mathcal{R}_{a_k}$ on the multiplicative side and the full new relation layer Λ_k on the relational side. Thus adding one atom adds exactly k current two-atom relations.

Chapter 14 — Combinatorial Amplification and Block Seams

Suppose k existing atoms form an old block O, and then m consecutive new atoms form a new block B.

Let

$$|O| = k, \qquad |B| = m.$$

All unordered two-atom relations split naturally as

$$\boxed{\Gamma(O \sqcup B) = \Gamma(O) \sqcup \Gamma(O,B) \sqcup \Gamma(B).} \tag{14.1}$$

where

$$|\Gamma(O)| = \frac{k(k+1)}{2},$$

$$|\Gamma(O,B)| = km,$$

$$|\Gamma(B)| = \frac{m(m+1)}{2}.$$

Theorem 14.1 — Combinatorial Amplification Law

After m consecutive new atoms are added, the exact number of new unordered two-atom relation instances is

$$\boxed{mk + \frac{m(m+1)}{2}.} \tag{14.2}$$

Proof

There are mk new–old relations.

Within the new block, self-pairing is allowed, so the number of internal relations is

$$\binom{m}{2} + m = \frac{m(m+1)}{2}.$$

The two classes are disjoint at the object level; adding them gives the result.

This proves the theorem.

14.1 A Relation Seam Is Not a Single Point

The cross-block relation set

$$\Gamma(O,B)$$

shows that the seam between old history and new atoms is not a single boundary.

It is an entire

$$k \times m$$

relation rectangle.

This observation naturally leads to richer notions of "seam," including square seams, generative seams, and phase seams. They share one basic fact:

When new structure enters the matrix, it adds not only positions but an entire layer of cross-boundary relations.

Chapter 15 — Persistent Embedding: Why Historical Relations Are Never Lost

The atomic sets satisfy

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

There are therefore natural embeddings at the level of relational objects.

Definition 15.1 — Layer Embedding

For $s \geq k$, define

$$\boxed{\iota_{k,s}: \Gamma_k \hookrightarrow \Gamma_s, \qquad \iota_{k,s}(\{a_i, a_j\}) = \{a_i, a_j\}.} \tag{15.1}$$

Theorem 15.2 — Persistent Inheritance of Lower-Order Relations

For every $s \geq k$, $\iota_{k,s}$ is injective, and every already formed relation instance retains the same endpoints and the same birth identity at every higher layer.

In particular,

$$\boxed{\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \Gamma_3 \hookrightarrow \cdots} \tag{15.2}$$

forms a monotone persistent system.

Proof

By definition,

$$\Gamma_k \subseteq \Gamma_s.$$

The map $\iota_{k,s}$ does not alter the relational object itself and is therefore plainly injective.

Neither the endpoints nor the birth layer changes.

This proves the theorem.

15.1 Confirmed Relations and Candidate Relations Must Be Distinguished

The set Γ_k in this part contains already confirmed atomic relations.

Relation sieves also introduce another kind of object:

candidate relations.

A candidate relation may be found, during sieving, to contain a composite endpoint and may then be eliminated.

But once a relation has entered

$$\Gamma_k,$$

it is a permanent part of history.

Therefore,

candidate relations may die;

confirmed relations only persist and are inherited.

(15.3)

This distinction is foundational for maintaining logical clarity in finite relation sieves.

Figure 1-10 Persistent Embedding Γ1→Γ2→Γ3→⋯

Confirmed historical relations are never resampled or deleted at higher layers; growth only adds new relations.

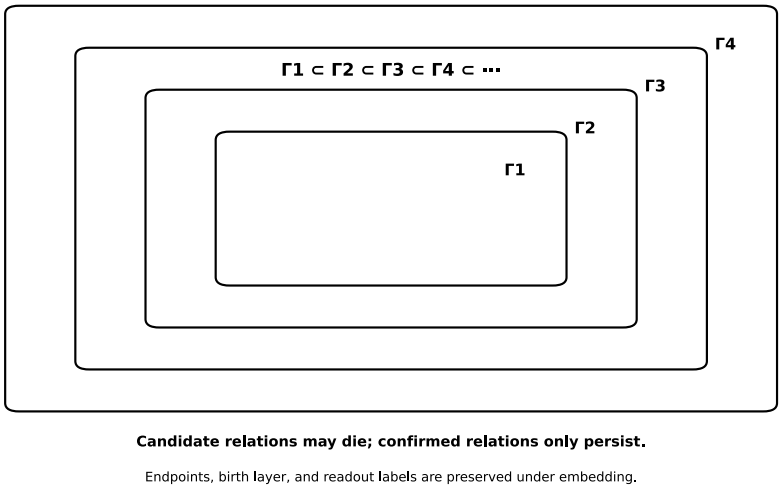


Figure 1-10. Persistent Embedding $\Gamma_1 \to \Gamma_2 \to \Gamma_3 \to \cdots$. The relation skeleton forms the monotone persistent system $\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \Gamma_3 \hookrightarrow \cdots$. Each higher layer only adds new relations; every confirmed relation preserves its endpoints, birth layer, and readout label.

Chapter 16 — Translation of Relation Rows: History Is Not Random Resampling

Let the gap between consecutive atoms be

$$\boxed{h_k = a_k - a_{k-1}.} \tag{16.1}$$

Fix a historical atom

$$b \in A_{k-1}.$$

Then

$$a_k \boxplus b = a_k + b + 1,$$

whereas

$$a_{k-1} \boxplus b = a_{k-1} + b + 1.$$

Subtracting gives

$$\boxed{(a_k \boxplus b) - (a_{k-1} \boxplus b) = h_k.} \tag{16.2}$$

Hence

$$\boxed{a_k \boxplus A_{k-1} = h_k + (a_{k-1} \boxplus A_{k-1}).} \tag{16.3}$$

Thus the historical part of the new relation row is an overall translation of the previous relation row, not a random regeneration.

The complete new relation layer also contains the self-pair

$$\{a_k, a_k\}.$$

A one-step relation-generation update may therefore be summarized as

$$\boxed{\text{permanent preservation of old history} + \text{global translation of the historical relation row} + \text{inserti}} \tag{16.4}$$

Chapter 17 — Lossless Sum–Difference Coordinates

The sum is important, but preserving only the sum is not relation-faithful.

Different atomic pairs may have the same sum.

Accordingly, each two-atom relation stores both its sum and difference.

Let

$$r = \{a_i, a_j\}, \qquad i \le j.$$

Definition 17.1 — Sum Label

Define

$$\boxed{\sigma(r) = a_i + a_j + 1.} \tag{17.1}$$

If

$$p_i = 2a_i + 1, \qquad p_j = 2a_j + 1,$$

then

$$\boxed{p_i + p_j = 2\sigma(r).} \tag{17.2}$$

Definition 17.2 — Difference Label

Define

$$\boxed{\delta(r) = a_j - a_i.} \tag{17.3}$$

Then

$$\boxed{p_j - p_i = 2\delta(r).} \tag{17.4}$$

Definition 17.3 — Sum–Difference Relation Coordinates

Define

$$\boxed{\Psi(r) = (\sigma(r), \delta(r)).} \tag{17.5}$$

Theorem 17.4 — Lossless Coordinate Theorem for Two-Atom Relations

Ψ is injective on its image.

If

$$\Psi(r) = (\sigma, \delta),$$

then the two endpoints are uniquely recovered as

$$\boxed{a_i = \frac{\sigma - 1 - \delta}{2}, \qquad a_j = \frac{\sigma - 1 + \delta}{2}.} \tag{17.6}$$

Proof

From

$$\sigma = a_i + a_j + 1$$

and

$$\delta = a_j - a_i,$$

adding and subtracting gives the formulas above.

Hence the same (σ, δ) cannot correspond to two distinct unordered endpoint pairs.

This proves the theorem.

17.1 Realizable Lattice Points

For an integer pair

$$(\sigma, \delta)$$

to correspond to an integer endpoint pair, it must at least satisfy

$$\sigma - 1 \geq \delta \geq 0,$$

and

$$\boxed{\sigma - 1 \equiv \delta \pmod 2.} \tag{17.7}$$

The endpoints are then recovered by

$$a_i = \frac{\sigma - 1 - \delta}{2}, \qquad a_j = \frac{\sigma - 1 + \delta}{2},$$

and one checks whether they belong to the atom set $\mathcal{A}$.

17.2 Fixed Sum and Fixed Difference Are Not Two Separate Worlds

Therefore,

$$\boxed{\text{fixed sum} \quad \text{and} \quad \text{fixed difference}}$$

are not two independent underlying relations.

They are simply two coordinate fibers of the same two-atom relation field.

This is the most direct reason that Goldbach and Twin can be returned to one common matrix.

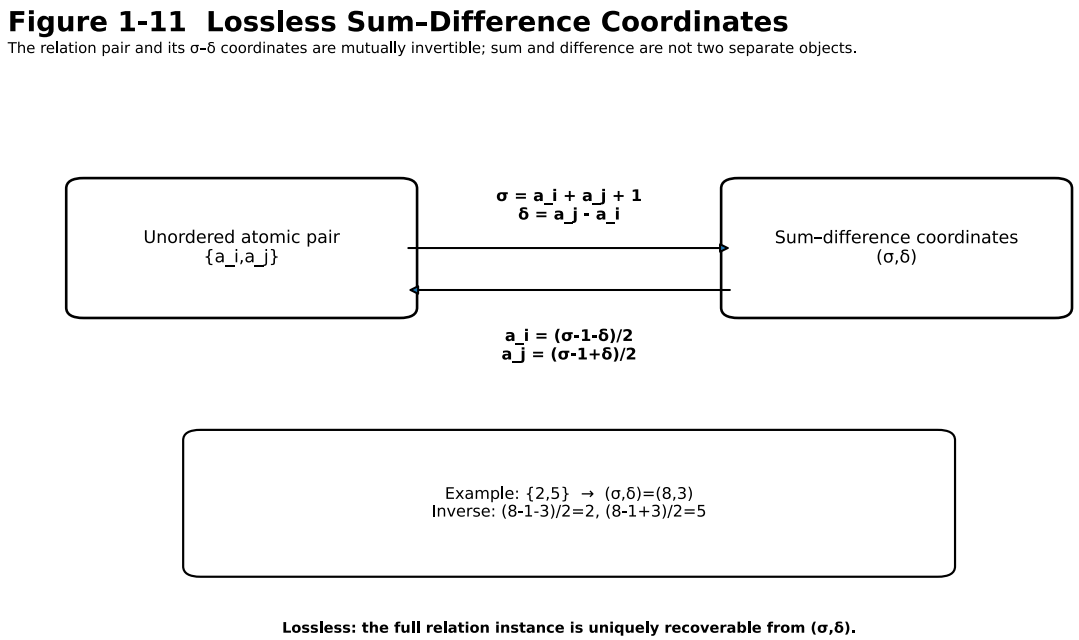


Figure 1-11. Lossless Sum–Difference Coordinates. The pair $\{a_i, a_j\}$ and the coordinates (σ, δ) are mutually invertible: $\sigma = a_i + a_j + 1$, $\delta = a_j - a_i$, and the two endpoints are uniquely recovered by the inverse formulas. Sum and difference are therefore two lossless readouts of the same relation object.

Chapter 18 — From "Relation Instances" to "Relation Slices"

The unified slicing language acts directly on the relation matrix.

The matrix supplies all existing relational objects.

A slice merely selects those satisfying a specified condition.

Definition 18.1 — Relation Readout

Let

$$\theta: \Gamma_\infty \to Y$$

be any relation label or relation coordinate.

For

$$y \in Y,$$

define the fiber

$$\boxed{\Gamma^{(\theta=y)} = \theta^{-1}(y).} \tag{18.1}$$

This operation creates no new relation instance; it only selects relations already present in the matrix.

Principle 18.2 — Inheritance Under Matrix-to-Subrelation Readout

If

$$r \in \Gamma^{(\theta=y)},$$

then the endpoints, birth layer, and identities of r are inherited entirely from Γ_∞.

Therefore,

Readout does not manufacture facts;
a subrelation inherits from the parent relation matrix.

(18.2)

Definition 18.3 — Fixed-Sum Slice

For $m \in \mathbb{N}$, define

$$\boxed{\Gamma^{(m)} = \{r \in \Gamma_\infty : \sigma(r) = m\}.} \tag{18.3}$$

Definition 18.4 — Fixed-Difference Slice

For $h \in \mathbb{N}_0$, define

$$\boxed{\Gamma^{[h]} = \{r \in \Gamma_\infty : \delta(r) = h\}.} \tag{18.4}$$

Thus:

- Goldbach-type fixed-sum problems read $\Gamma^{(m)}$;
- Twin-type fixed-gap problems read $\Gamma^{[1]}$;
- general fixed-gap problems read $\Gamma^{[h]}$.

These are only the first two examples of the relation-slice language.

Chapter 19 — From Two-Atom Slices to General Position Maps

Fixed sums and fixed differences can both be expressed directly inside Γ_∞.

Modern prime relations, however, extend far beyond two-atom sums and differences.

Examples include

$$p \mapsto 2p + 1,$$

$$n \mapsto n^2 + 1,$$

$$a \mapsto (a, a + d, \dots, a + (k-1)d),$$

and the window

$$(n^2, (n+1)^2).$$

To unify these relations, "slicing" must be extended from coordinate fibers in Γ_∞ to general readouts of the fixed integer spectrum.

Definition 19.1 — Family of Position Maps

Let the parameter be Θ, with anchor space X_Θ.

Given finitely many integer-valued maps

$$\boxed{f_i : X_\Theta \times \Theta \to \mathbb{Z}, \qquad 1 \le i \le m,} \tag{19.1}$$

call

$$(f_1, \dots, f_m)$$

a **family of position maps**.

Definition 19.2 — Local Relation Predicate

After choosing fields of the fixed spectrum, define a finite logical function

$$\mathcal{B}$$

on the corresponding positional readings.

The most basic prime-bundle predicate is

$$\boxed{R_\Theta(x) = \prod_{i=1}^{m} \alpha\,(f_i(x;\Theta)).} \tag{19.2}$$

Then

$$R_\Theta(x) = 1$$

if and only if all positions are prime.

19.1 A Fixed Sum as a Position Map

Given an even integer N,

$$f_1(p;N) = p, \qquad f_2(p;N) = N - p.$$

Hence

$$\boxed{R_N(p) = \alpha(p)\alpha(N-p).} \tag{19.3}$$

19.2 A Fixed Difference as a Position Map

Given d,

$$f_1(p) = p, \qquad f_2(p) = p + d.$$

Hence

$$\boxed{R_d(p) = \alpha(p)\alpha(p+d).} \tag{19.4}$$

19.3 A Finite Pattern as a Bundle of Position Maps

Given

$$H = \{h_1, \dots, h_k\},$$

define

$$f_j(n) = n + h_j.$$

Then

$$\boxed{R_H(n) = \prod_{j=1}^{k} \alpha\,(n + h_j).} \tag{19.5}$$

19.4 An Arithmetic-Progression Bundle as a Parameterized Pattern

Given k, d,

$$H_{k,d} = \{0, d, 2d, \dots, (k-1)d\}.$$

Then

$$\boxed{R_{k,d}(a) = \prod_{j=0}^{k-1} \alpha\,(a + jd).} \tag{19.6}$$

19.5 Affine Relations

Given A, B,

$$f_1(p) = p, \qquad f_2(p) = Ap + B.$$

Then

$$\boxed{R_{A,B}(p) = \alpha(p)\alpha(Ap+B).} \tag{19.7}$$

The Sophie Germain relation corresponds to

$$A = 2, \qquad B = 1.$$

19.6 Polynomial Pullback

Given an integer-valued polynomial

$$f(n),$$

define

$$\boxed{R_f(n) = \alpha(f(n)).} \tag{19.8}$$

For simultaneous readout of several polynomials,

$$\boxed{R_{\boldsymbol{f}}(n) = \prod_{j=1}^{k} \alpha\left(f_j(n)\right).} \tag{19.9}$$

Chapter 20 — Windows, Consecutive Primes, and Complete Factor Identity Are Also Slices

A slice does not require every problem to be written as "several fixed positions are simultaneously prime."

As long as the readout objects still come from the same fixed spectrum, they belong to the same underlying language.

20.1 Window Occupancy

Given a finite integer window

$$I_\theta \subset \mathbb{Z},$$

define

$$\boxed{W(\theta) = \sum_{n \in I_\theta} \alpha\,(n).} \tag{20.1}$$

Then

$$W(\theta) > 0$$

means that the window contains at least one prime.

For example, the Legendre window is

$$I_n = (n^2, (n+1)^2) \cap \mathbb{Z}.$$

20.2 Empty-Window Readout

Define

$$\boxed{E(I) = \prod_{n \in I} (\,1 - \alpha(n)).} \tag{20.2}$$

Then

$$E(I) = 1$$

if and only if I contains no prime.

Polignac's condition "both endpoints are prime and there is no prime in between" is exactly

$$\alpha(p)\alpha(p+d) \cdot \prod_{p<n<p+d} (\,1 - \alpha(n)).$$

20.3 Indexing Consecutive Primes

If

$$p_n$$

denotes the n-th prime, then a consecutive-prime relation reads

$$(p_n, p_{n+1}).$$

For example, the local Andrica predicate is

$$\boxed{\sqrt{p_{n+1}} - \sqrt{p_n} < 1.} \tag{20.3}$$

Firoozbakht can be written as the stable logarithmic comparison

$$\boxed{\frac{\log p_{n+1}}{n+1} < \frac{\log p_n}{n}.} \tag{20.4}$$

The object being read is still the fixed prime sequence; only the relation predicate has been upgraded from a Boolean primality test to a real-valued comparison.

20.4 Complete Factor-Set Slices

A Grimm-type relation reads

$$\mathrm{PF}(n+i)$$

and requires distinct prime factors to be chosen for the positions in a block of consecutive composite integers.

An Erdős–Woods-type relation compares

$$\mathrm{PF}(x+i)$$

with

$$\mathrm{PF}(y+i)$$

and asks whether they agree position by position.

This shows that the matrix of Unified Prime Relations Theory is not restricted to

$$\alpha(n) \in \{0,1\}.$$

Richer identity fields belong to the same fixed spectrum.

Chapter 21 — Unified Relation Normal Form

Compressing the preceding structures gives one of the most important relational expressions in the entire book.

Definition 21.1 — Fixed-Spectrum Relation Normal Form

A finite local relation is written as

$$\boxed{R_\Theta(x) = \mathcal{B}(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_m(x;\Theta))),} \tag{21.1}$$

where:

- Θ is the parameter;
- x is the anchor;
- f_i are position maps;
- $\boldsymbol{I}$ supplies fixed identity fields;
- $\mathcal{B}$ is a finite relation predicate.

If only primality identity is read, this simplifies to

$$\boxed{R_\Theta(x) = \mathcal{B}(\alpha(f_1(x;\Theta)), \dots, \alpha(f_m(x;\Theta))).} \tag{21.2}$$

21.1 A Proposition Adds Quantifiers Only at the Outermost Layer

A concrete mathematical proposition is then written as

$$\boxed{\mathcal{Q}_{\Theta,x}[R_\Theta(x)].} \tag{21.3}$$

Thus differences among classical problems decompose into three levels:

$$\boxed{\text{position map} + \text{local predicate} + \text{outer quantifier.}} \tag{21.4}$$

This is the complete mathematical meaning of a “slice.”

Figure 1-12 Unified Relation Normal Form

Classical propositions vary through position maps, fixed identity fields, a local predicate, and an outer quantifier.

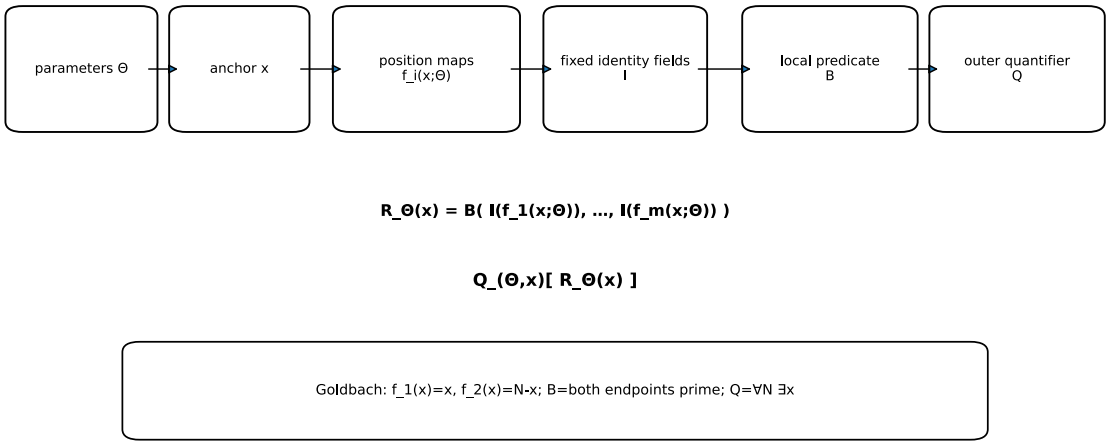


Figure 1-12. Unified Relation Normal Form. A finite local relation is written uniformly as $R_\Theta(x) = \mathcal{B}(\boldsymbol{I}(f_1(x;\Theta)), \ldots, \boldsymbol{I}(f_m(x;\Theta)))$, while a concrete proposition adds only the outer quantifier $\mathcal{Q}_{\Theta,x}[R_\Theta(x)]$. This cleanly separates position maps, identity readout, the local predicate, and the global quantifier.

Chapter 22 — Relation Fidelity: Why One Cannot Compress Too Early to a Single Number

Under the unified representation, another question immediately arises:

> *Can a relation retain only "how many remain" without retaining "which ones remain"?*

In general, the answer is no.

For a fixed-sum problem, for example, the genuine object is not merely a candidate count but a concrete family of reflection slots

$$(x, N - x).$$

Two distinct targets may have the same number of candidates at one sieve layer, yet because the remaining positions have different residue-class configurations, their updates at the next layer can differ.

Therefore,

$$\boxed{\text{equal counts} \Rightarrow/ \text{ equal relation states.}} \tag{22.1}$$

This gives an underlying principle that relation sieves must obey:

Preserve the relational objects first; only then decide which statistics may be compressed.

(22.2)

22.1 Minimal Information in a Relation State

For a relation slot to remain traceable during a sieve process, one must at least be able to record

$$\boxed{\text{position} + \text{endpoints} + \text{endpoint identities} + \text{the first killer when needed} + \text{birth and death histo}} \tag{22.3}$$

A relation-faithful sieve is the systematic form of this principle.

This part fixes only its logical foundation:

> *A relation is an object, not an explanatory appendix to a number.*

Chapter 23 — The Natural Entry Point to Higher-Order Relations

Two-atom relations are not the upper limit of the matrix.

For any $r \geq 2$, define at layer k the family of unordered r-atom relations

$$\boxed{\Gamma_k^{(r)} = \{\{a_{i_1}, \dots, a_{i_r}\}: 1 \leq i_1 \leq \cdots \leq i_r \leq k\}.} \tag{23.1}$$

Whether repeated endpoints are allowed can be decided by the definition of the particular relation.

When a new atom a_k is born, every new higher-order relation containing a_k is generated at the same time.

Thus the binary idea of "full-row birth" generalizes to

> one new atom generates one complete new star-layer of relations.

(23.2)

This gives a unified object-level entry point for

- ternary Goldbach;
- prime k-tuples;
- Green–Tao arithmetic-progression chains;
- general finite reflection bundles.

Chapter 24 — Formal Definition of the Finite-Layer Matrix

The atomic generative structure and the fixed-spectrum structure can be combined into one unified object.

Definition 24.1 — Generative Relation Matrix at Layer k

Define

$$\boxed{\mathfrak{M}_k = \left(\boldsymbol{I}, A_k, \mathcal{M}_k, \Gamma_k, \{\Lambda_j\}_{j\le k}, \Phi, \star, \boxplus, \Psi\right).} \tag{24.1}$$

where:

1. $\boldsymbol{I}$ is the fixed integer identity spectrum;
2. $A_k = \{a_1, \dots, a_k\}$ is the set of generated odd-prime atoms;
3. $\mathcal{M}_k = \bigcup_{i\le k} \mathcal{R}_{a_i}$ is the multiplicative occupancy field generated by existing atoms;
4. Γ_k is the complete two-atom relation skeleton;
5. Λ_j is the relation-birth layer at stage j;
6. Φ is the odd-number coordinate map;
7. $\star$ is the multiplicative coordinate operation;
8. $\boxplus$ is the additive relation operation for two odd integers;
9. $\Psi = (\sigma, \delta)$ gives lossless sum–difference coordinates for two-atom relations.

Definition 24.2 — Infinite Relation Matrix

Using the natural embeddings

$$\mathfrak{M}_k \hookrightarrow \mathfrak{M}_{k+1},$$

define

$$\boxed{\mathfrak{M}_\infty = \varinjlim_{k\to\infty} \mathfrak{M}_k.} \tag{24.2}$$

On the two-atom relation skeleton,

$$\boxed{\Gamma_\infty = \bigcup_{k\ge 1} \Gamma_k.} \tag{24.3}$$

The direct limit is more precise than a bare union because it also records the historically compatible embeddings between layers.

Chapter 25 — Zeroth-Order Closure of Matrix Dynamics

The preceding definitions and theorems give a complete finite recursive loop:

$$
\begin{array}{ccc}
 & A_k & \\
 & \swarrow \quad \searrow & \\
\text{multiplicative channel} & & \text{relational channel} \\
\downarrow & & \downarrow \\
\mathcal{M}_k & & \Gamma_k \\
\downarrow & & \downarrow \\
\text{find the least escape point} & & \text{preserve all historical relations} \\
\downarrow & & \downarrow \\
a_{k+1} & & \Gamma_k \\
 & \searrow \quad \swarrow & \\
 & \Gamma_{k+1} = \Gamma_k \sqcup \Lambda_{k+1} &
\end{array}
\tag{25.1}
$$

Compressed into one sentence:

$$
\boxed{\text{multiplicative occupancy} \rightarrow \text{irreducible release} \rightarrow \text{new atom} \rightarrow \text{full-row birth of relations} \rightarrow \text{per}} \tag{25.2}
$$

No particular classical conjecture has been used here.

Accordingly, a concrete conjecture appears only as a readout of

$$\mathfrak{M}_{\infty},$$

and does not define the matrix in reverse.

Chapter 26 — A Complete Small Example

The first four odd primes are

$$3,5,7,11.$$

The corresponding atoms are

$$a_1 = 1, \qquad a_2 = 2, \qquad a_3 = 3, \qquad a_4 = 5.$$

Thus

$$A_4 = \{1,2,3,5\}.$$

26.1 Fixed Integer Identity Spectrum

On the ordinary integer axis,

$$\mathfrak{I}(3) = A,$$

$$\mathfrak{I}(5) = A,$$

$$\mathfrak{I}(7) = A,$$

$$\mathfrak{I}(9) = \mathcal{M}_3,$$

$$\mathfrak{I}(11) = A,$$

$$\mathfrak{I}(15) = \mathcal{M}_3.$$

These identities do not change according to whether one later chooses a sum or a difference relation.

26.2 Atomic Rays

The atom 1 corresponds to the prime 3, and its ray is

$$\mathcal{R}_1 = \{4,7,10,13, \dots\},$$

because it has the form

$$1 + 3b.$$

Mapping back to the odd-integer axis gives

$$\Phi(4) = 9, \qquad \Phi(7) = 15, \qquad \Phi(10) = 21.$$

All are divisible by the prime 3.

The atom 2 corresponds to the prime 5, with ray

$$\mathcal{R}_2 = \{7,12,17,22, \dots\}.$$

The atom 3 corresponds to the prime 7, with ray

$$\mathcal{R}_3 = \{10,17,24,31, \dots\}.$$

Coordinate 4 is occupied by an existing ray and corresponds to the odd composite 9.

Coordinate 5 is not occupied by any of the first three atomic rays, so it becomes the next atom, corresponding to the prime 11.

26.3 Relation-Birth Layers

$$\Lambda_1 = \{\{1,1\}\},$$

$$\Lambda_2 = \{\{1,2\},\{2,2\}\},$$

$$\Lambda_3 = \{\{1,3\},\{2,3\},\{3,3\}\},$$

$$\Lambda_4 = \{\{1,5\},\{2,5\},\{3,5\},\{5,5\}\}.$$

Therefore,

$$|\Gamma_4| = 1 + 2 + 3 + 4 = 10.$$

26.4 Why a Sum Collision Cannot Be Used to Compress Away the Relation

We have

$$1 \boxplus 3 = 5,$$

and

$$2 \boxplus 2 = 5.$$

Thus the two distinct relations

$$\{1,3\}, \qquad \{2,2\}$$

both project to 5 on the sum axis.

But their lossless coordinates differ:

$$\Psi(\{1,3\}) = (5,2),$$

$$\Psi(\{2,2\}) = (5,0).$$

Therefore,

$$\boxed{\text{same sum} \not\Rightarrow \text{same relation.}} \tag{26.1}$$

This is why the matrix must preserve relation instances, not merely sum support.

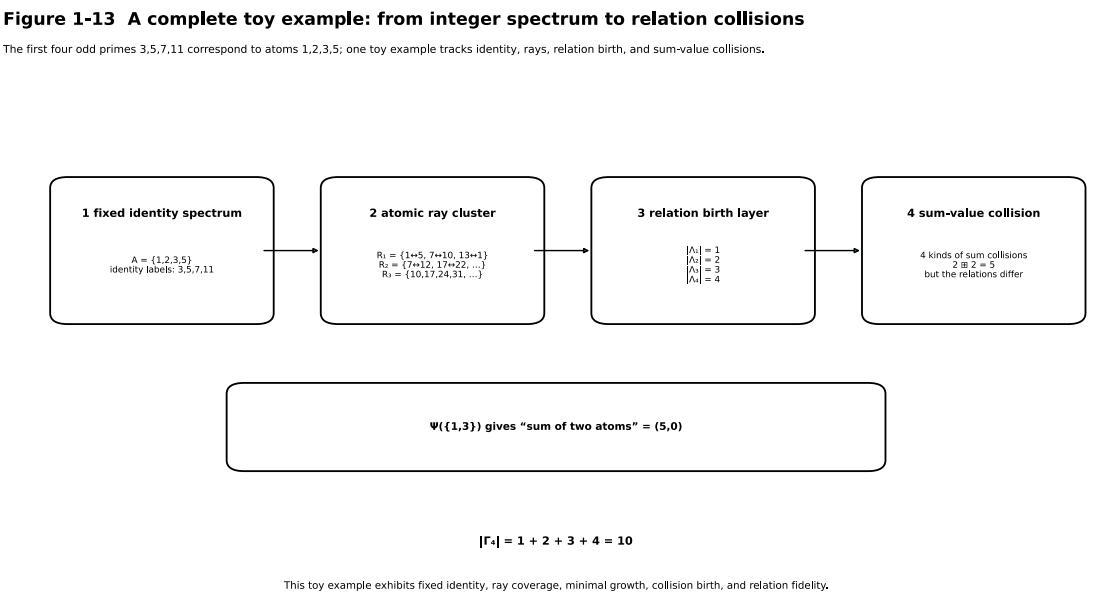


Figure 1-13. A Complete Small Example: From the Integer Spectrum to Relation Collisions. The first four odd primes 3,5,7,11 correspond to atoms 1,2,3,5. The same small example exhibits fixed integer identity, atomic rays, relation-birth layers, and a sum collision; {1,3} and {2,2} have the same sum but different lossless coordinates, showing why relation instances cannot be compressed into a sum count.

Chapter 27 — Summary Table of Foundational Theorems in Part I

For uniform reference throughout the book, the results of this part are compressed into the following foundational theorems.

Foundational Theorem A — Fixed Identity

Every integer position n has a fixed arithmetic identity

$$\boldsymbol{I}(n),$$

independent of whatever relation is chosen later.

Foundational Theorem B — Multiplicative Compression

$$\boxed{\Phi(a \star b) = \Phi(a)\Phi(b).}$$

Foundational Theorem C — Atomic Equivalence

$$\boxed{a \in \mathcal{A} \Leftrightarrow \Phi(a) \text{ is an odd prime.}}$$

Foundational Theorem D — Recursive Generation of Atoms

$$\boxed{a_{k+1} = \min\{n > a_k : n \notin \mathcal{M}_k\}.}$$

Foundational Theorem E — Full-Row Birth of Relations

$$\boxed{\Gamma_k = \Gamma_{k-1} \sqcup \Lambda_k, \qquad |\Lambda_k| = k.}$$

Foundational Theorem F — Persistent Inheritance

$$\boxed{\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \cdots.}$$

Foundational Theorem G — Translation of Relation Rows

$$\boxed{a_k \boxplus A_{k-1} = h_k + (a_{k-1} \boxplus A_{k-1}).}$$

Foundational Theorem H — Lossless Sum–Difference Coordinates

$$\boxed{\Psi(r) = (\sigma(r), \delta(r))}$$

uniquely determines the endpoints of a two-atom relation.

Foundational Theorem I — Relation-Slicing Principle

Every classical relation first specifies only position maps and a local predicate:

$$\boxed{R_\Theta(x) = \mathcal{B}(\boldsymbol{I}(f_1(x;\Theta)), \ldots, \boldsymbol{I}(f_m(x;\Theta))).}$$

A relation does not alter the fixed identity spectrum.

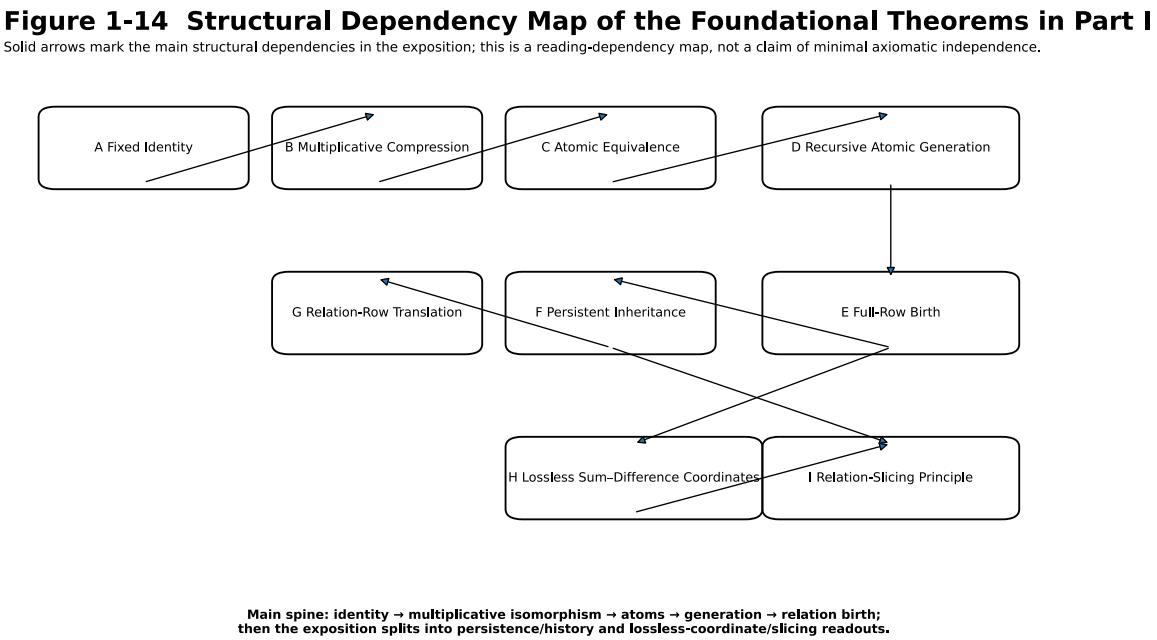


Figure 1-14. Structural Dependency Map of the Foundational Theorems in Part I. The foundational theorems A–I of Part I form a structural chain from fixed identity, multiplicative compression, atomic equivalence, and recursive generation to full-row relation birth, persistent inheritance, relation-row translation, lossless sum–difference coordinates, and relation slicing. The diagram records structural/expository dependence, not minimal axiomatic independence.

Chapter 28 — From Relation Slices to Quantifier Structures

This part has answered four questions:

1. where integer identity resides;
2. why odd primes are multiplicatively irreducible atoms;
3. how new atoms are recursively generated;
4. how relations among atoms form a persistent matrix and can be sliced in different ways.

Classical prime-number problems, however, often impose stronger outer requirements.

For example,

$$\forall N\ \exists p,$$

or

$$\exists^{\infty} p,$$

or

$$\forall k\ \exists a, d,$$

or

$$\forall n\ \exists p \in I_n.$$

These ask not merely "what is the relation?" but rather

> how the relation continues to hold across an entire parameter family.

On this foundation, the rest of the book unfolds layer by layer:

finite horizons → relation histories → quantifier types → local obstruction → seams and periods (28.1)

In particular,

> **Finite verification is not itself an infinite quantifier;**
>
> **a finite structure enters a universal or infinitary conclusion only through an explicit closure mechanism.**

(28.2)

This forms the core of the middle part of the book.

Chapter 29 — How to Read the Relation Matrix

The entire structure can be viewed as an ever-unfolding graph whose underlying identities never move.

On the integer line, every position already has its arithmetic identity.

Odd coordinates reveal multiplicative irreducibility as atoms.

At the birth of an atom, it immediately forms relations with all historical atoms.

These relations are permanently preserved.

They can then be read in different ways:

- along the sum axis;
- along the difference axis;
- along affine lines;
- along finite patterns;
- along arithmetic-progression bundles;
- through windows;
- along polynomial orbits;
- through consecutive-prime indices;
- through complete factor sets.

Thus the first global diagram of Unified Prime Relations Theory is

$$\boxed{\begin{array}{c}\text{fixed integer identity spectrum } \boldsymbol{I}\\ \Downarrow\\ \text{atomic coordinates for odd multiplication } (\mathbb{N}_0,\star)\\ \Downarrow\\ \text{irreducible atoms } \mathcal{A}\\ \Downarrow\\ \text{atomic rays and least escape}\\ \Downarrow\\ \text{relation-birth layers } \Lambda_k\\ \Downarrow\\ \text{persistent relation matrix } \Gamma_\infty\\ \Downarrow\\ \text{position maps } f_i\\ \Downarrow\\ \text{relation slices } R_\Theta\\ \Downarrow\\ \text{outer quantifiers.}\end{array}} \tag{29.1}$$

Part I establishes the theory up through “relation slices.”

The strict classification of quantifiers, and the mechanisms by which finite structures yield universal or infinite conclusions in batch, form the core of the subsequent quantifier theory.

Conclusion — From a List of Primes to a Readable Relation Spectrum

Traditional exposition often writes the primes as a sequence

$$2,3,5,7,11,13,\ldots$$

and then begins asking questions.

Unified Prime Relations Theory adopts a different order of organization.

It first acknowledges:

integer identities are already fixed.

It then transports odd multiplication to the shared coordinate system and sees that

odd primes are multiplicatively irreducible atoms.

Then it sees that

a new atom does not merely add one point;
it simultaneously adds an entire new layer of relations.

Once these relations enter history, they are permanently preserved:

$$\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \cdots.$$

Fixed sums and fixed differences are no longer two separate underlying structures:

$$\Psi(r) = (\sigma(r), \delta(r))$$

already shows that they are different coordinates of the same relational object.

More generally, affine relations, windows, arithmetic progressions, polynomials, consecutive primes, and complete factor sets do not require new matrices to be built from scratch.

They need only specify

where to read

and

how to combine those readings.

Thus what Part I ultimately establishes is not merely an "atomic matrix," but a richer fixed relation spectrum:

One fixed identity spectrum permits infinitely many relation slices;
the slices may differ, while the underlying identities never move.

This is also the first principle used throughout the rest of the book:

The matrix provides identity;
relations choose the readout;
propositions specify the quantifier.

Part II studies finite horizons and relation histories.

It asks:

When can a concrete finite relation inside an infinite matrix acquire a complete, zero-error, irreversible terminal identity?

A further question is:

How do different universal and infinite quantifiers close in batch from finite structures?

Within this framework, Goldbach, Twin, Green–Tao, Legendre, Mersenne, Grimm, and more general user-defined relations can all enter the same rigorous mathematical language as different slices of one fixed spectrum.

Part II — Finite Moving Horizons, Relation Reconstruction, and Permanent Preservation

— Persistent Inheritance, Terminal Certification, Relation Histories, and the Finite Entry Point to Infinity

Purpose of This Part

The fixed arithmetic identity spectrum gives the factual identity of integers, but it does not automatically give relations.

Whether a fixed sum is occupied, whether a fixed gap recurs, whether a window is nonempty, or whether a pattern is hit simultaneously depends on whether these fixed identities can be correctly reorganized according to the specified relation.

Therefore:

The fixed spectrum is the factual foundation; relation reconstruction is the structural condition for a proposition to hold.

Moreover, because global relations usually contain universal or infinitary quantifiers, every rigorous argument must answer a second question:

Why can an infinitely growing prime-relation matrix be rigorously verified through finite information?

The central answer is:

A finite moving horizon is not an approximate truncation of an infinite relation, but the finite entry point through which an

Abstract

Part I established the fixed arithmetic identity spectrum

$$\boldsymbol{I}(n)$$

and the atomic generative structure in the shared odd-number coordinate system. Odd primes correspond to $\star$-irreducible atoms; once the k-th new atom a_k is born, it immediately forms the new relation layer

$$\Lambda_k$$

with all historical atoms, yielding the persistent embeddings

$$\Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \cdots.$$

These results provide a unified relation matrix, but fixed identity alone is not equivalent to solving a concrete relation. A relation must preserve position, pairing, window, order, or other structural information; if it is compressed too early into a total count, density, or finite-modulus histogram, the fixed spectrum cannot recover pairing information that has already been discarded.

This part first establishes the **principle of relation reconstruction**. A relation state is called relation-faithful if its record allows the relation endpoints and their fixed identities to be reconstructed. For low-order relations such as fixed sums, fixed gaps, and finite patterns, relational objects must be preserved before scalar statistics. An explicit fixed-sum example shows that two targets can have the same initial number of slots and the same number of survivors after the first sieve layer, yet undergo different updates at the next layer. Thus “the number remaining” is not a sufficient description of the relation state.

Next, a unified **finite moving horizon** is established. For any finite upper bound X, define

$$\mathcal{V}(X) = \{a \geq 1 : \Phi(a) \leq X\}.$$

The atomic identity of any position in this window can be decided completely using only already generated atomic layers not exceeding $\sqrt{X}$, because every odd composite not exceeding X has a prime factor not exceeding $\sqrt{X}$. This yields zero-error terminal certification:

finite window + atomic layers up to the square root ⇒ all atomic identities in the window are terminally determined.

Consecutive prime squares

$$[p_k^2, p_{k+1}^2)$$

provide the natural discrete boundaries of this moving horizon: every composite identity in the whole interval is completely determined by historical atoms no larger than p_k.

This part strictly distinguishes two kinds of objects:

1. **candidate relation shells**: their endpoints have not yet all completed atomic certification and may be eliminated at later multiplicative layers;
2. **permanent relation skeletons**: their endpoints have been confirmed as atoms, and once they enter the relation matrix they are preserved permanently.

For each odd composite, the least prime factor defines its unique genuine death layer; for a finite relation, the least prime factor that first certifies any endpoint as composite defines the unique relation-death layer. Thus every finite candidate relation either undergoes genuine elimination at some finite layer or completes certification in a finite moving horizon and is promoted to a permanent relation. No finite relation remains unresolved forever.

The matrix dynamics therefore has the clear form

$$\text{candidate birth} \rightarrow \begin{cases} \text{genuine elimination,} \\ \text{terminal certification and promotion} \end{cases} \rightarrow \text{permanent persistence.}$$

On this basis, the part defines relation histories recording supporting endpoints, birth layer, genuine death layer, terminal horizon, readout labels, and final state, and proves that all confirmed relations remain unchanged under horizon expansion.

Finally, finite terminal structure is connected to universal and infinitary quantifiers. If the finite frozen layers of a relation class satisfy persistent embedding, then

the global relation is infinite ⇔ there is no last moving horizon that produces a new frozen relation.

Different quantifiers have different exact finite entry forms. Infinite recurrence becomes unbounded cumulative count; parameterwise universality becomes nonemptiness of every relation fiber; window universality becomes occupancy of every window; arbitrary-length relations become unbounded realized maximum length.

Thus the fixed spectrum does not weaken relational problems. On the contrary, only after relational objects, finite terminal structure, and quantifier structure have been reconstructed layer by layer can the fixed spectrum uniformly carry different prime-number problems.

Keywords: relation reconstruction; finite moving horizon; relation fidelity; permanent relation; terminal certification; genuine death; relation history; square horizon; persistent embedding; universal quantifier; infinite relation

Chapter 1 — Why Fixed Identity Cannot Replace Relation

The arithmetic identity of an integer is fixed.

For example, if

$$29 \in \mathbb{P},$$

then whether one is discussing Goldbach, Twin, arithmetic progressions, or polynomial relations,

$$\alpha(29) = 1$$

does not change.

But a relational proposition asks not only "what is a given position?" It also asks:

> Which positions must occur simultaneously, how do they correspond, and do they lie in the same relation fiber?

For example, the binary fixed-sum relation for an even integer N requires

$$\boxed{\exists p, \qquad \alpha(p)\alpha(N-p) = 1.} \tag{1.1}$$

The key issue is not merely knowing which integers are prime, but preserving the reflection map

$$\boxed{p \mapsto N - p.} \tag{1.2}$$

If one keeps only

$$\pi(N),$$

or the number of primes in an interval, or even histograms of all primes modulo several small primes, that information generally cannot reconstruct the prescribed reflection pairing.

Therefore,

$$\boxed{\text{fixed identity} \neq \text{relational fact.}} \tag{1.3}$$

More precisely,

$$\boxed{\text{fixed identity} + \text{correct relation reconstruction} = \text{concrete relational fact.}} \tag{1.4}$$

This is also the strict meaning of the word "slice" throughout the book.

A slice is not an arbitrary observation of the fixed spectrum.

It must carry enough information for the original relation to be identified again from the slice record.

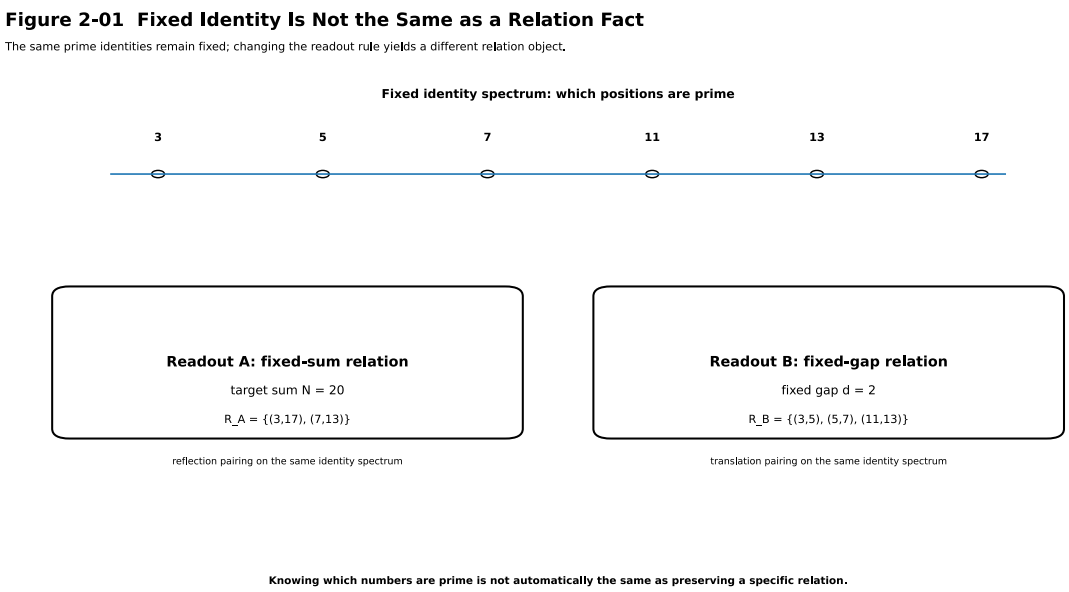


Figure 2-01. Fixed Identity Is Not the Same as a Relation Fact. The fixed spectrum tells us which positions have prime identity; a concrete relation must additionally specify how endpoints are paired. The same primes may form reflection pairings under a fixed sum or translation pairings under a fixed gap. Identity facts therefore cannot replace relation objects.

Chapter 2 — Relation Reconstruction: What It Means to Truly Preserve a Relation

Definition 2.1 — Relation Instance

Let

$$\Theta$$

be a parameter,

$$x$$

an anchor, and

$$f_1, \dots, f_m$$

position maps.

Define the relation instance

$$\boxed{r_{\Theta,x} = (f_1(x;\Theta), \dots, f_m(x;\Theta)).} \tag{2.1}$$

Its fixed-identity reading is

$$\boxed{\boldsymbol{I}(r_{\Theta,x}) = (\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_m(x;\Theta))).} \tag{2.2}$$

Definition 2.2 — Relation Descriptor

Let

$$D(r)$$

be some record of a relation instance r.

If there exists a deterministic reconstruction map

$$\mathcal{R}$$

such that

$$\boxed{\mathcal{R}(D(r)) = r,} \tag{2.3}$$

then D is called a **lossless relation descriptor**.

Definition 2.3 — Target-Sufficient Descriptor

Given a local relation predicate

$$\mathcal{B}(r) \in \{0,1\},$$

if there exists a function F such that

$$\boxed{\mathcal{B}(r) = F(D(r))} \tag{2.4}$$

for all relation instances, then D is called **target-sufficient** for that predicate.

Every lossless descriptor is target-sufficient; a target-sufficient descriptor need not be lossless.

This distinction is important.

Some problems require only a local truth-value decision and can use a smaller sufficient state; others must track birth, death, endpoint pairing, or later updates and therefore require richer relational objects.

2.1 Sum–Difference Coordinates Are Lossless

Part I defined

$$\Psi(r) = (\sigma(r), \delta(r))$$

and proved

$$a_i = \frac{\sigma - 1 - \delta}{2}, \qquad a_j = \frac{\sigma - 1 + \delta}{2}.$$

Hence Ψ is a lossless relation descriptor.

2.2 The Sum Alone Is Not Lossless

If one preserves only

$$\sigma(r),$$

different relations may project to the same sum.

For example,

$$1 \boxplus 3 = 2 \boxplus 2 = 5.$$

Therefore sum support cannot reconstruct the original relation.

2.3 This Is Exactly Where the Value of Relation Theory Lies

The fixed spectrum solves

positional identity.

Relation theory solves

how positions are organized into objects.

Therefore,

Relations are not annotations attached to the fixed spectrum;
relations are the structure that organizes fixed facts into propositions.

(2.5)

The power of unification comes from studying the relation layer deeply enough, not merely from the fact that “the spectrum is fixed.”

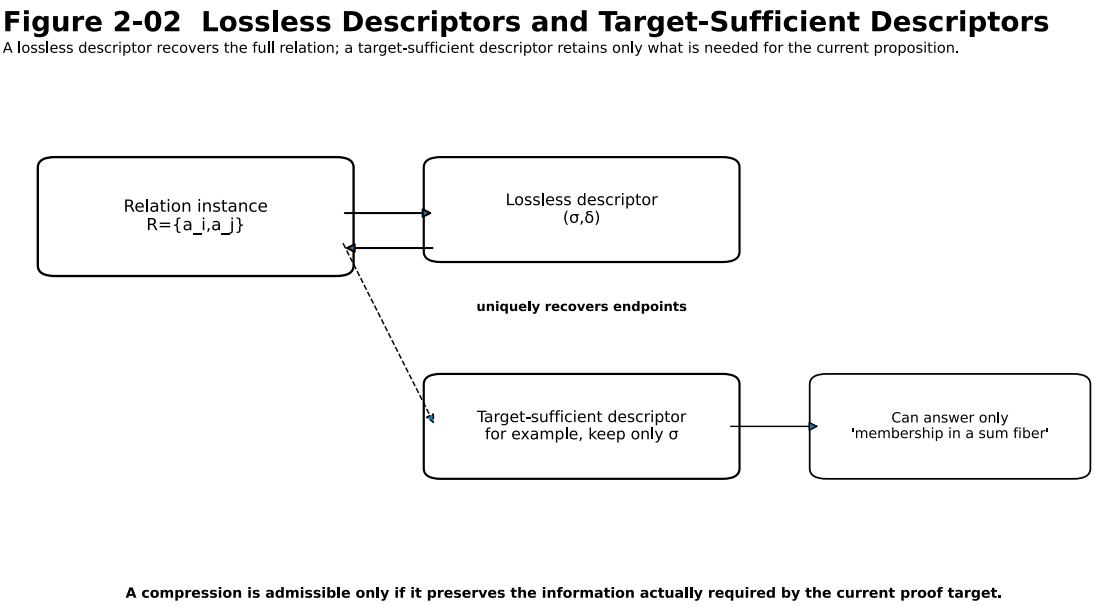


Figure 2-02. Lossless Descriptors and Target-Sufficient Descriptors. A lossless descriptor must reconstruct the full relation instance, whereas a target-sufficient descriptor need retain only the information required by the current target. For example, (σ, δ) reconstructs the two atomic endpoints, while σ alone can test fixed-sum-fiber membership but cannot recover the full relation.

Chapter 3 — Persistent Relation Filtering: Why Old Relations Are Never Lost

Let the odd primes be ordered as

$$3 = p_1 < p_2 < p_3 < \cdots,$$

with corresponding reduced atoms

$$a_k = \frac{p_k - 1}{2}.$$

Write

$$A_k = \{a_1, \ldots, a_k\}.$$

Part I defined the two-atom relation layer

$$\Gamma_k = \big\{\{a_i, a_j\} : 1 \le i \le j \le k\big\}.$$

At the birth of the k-th atom, the new relation layer is

$$\Lambda_k = \{\{a_i, a_k\} : 1 \le i \le k\},$$

with

$$\boxed{\Gamma_k = \Gamma_{k-1} \sqcup \Lambda_k.} \tag{3.1}$$

Definition 3.1 — Relation Birth Layer

If

$$r = \{a_i, a_j\}, \qquad i \le j,$$

define

$$\boxed{b(r) := j.} \tag{3.2}$$

Thus the relation r first enters the matrix when its later endpoint a_j is born.

Theorem 3.2 — Unique Birth-Layer Theorem

Every confirmed two-atom relation

$$r \in \Gamma_\infty$$

has a unique birth layer

$$b(r),$$

and

$$r \in \Lambda_{b(r)}.$$

Proof

Let

$$r = \{a_i, a_j\}, \qquad i \le j.$$

At layer $j-1$, the atom a_j has not yet appeared, so

$$r \notin \Gamma_{j-1}.$$

At layer j, a_j is born and forms relations with every historical atom

$$a_1, \dots, a_j.$$

Hence

$$r \in \Lambda_j \subset \Gamma_j.$$

Since the layers Λ_j are pairwise disjoint, the birth layer is unique.

This proves the theorem.

Theorem 3.3 — Persistent Inheritance of Lower-Order Relations

If

$$s \ge t,$$

there is a natural embedding

$$\iota_{t,s}: \Gamma_t \hookrightarrow \Gamma_s$$

such that for every

$$r \in \Gamma_t,$$

we have

$$\iota_{t,s}(r) = r.$$

Therefore,

$$\boxed{\Gamma_t \subseteq \Gamma_s.} \tag{3.3}$$

Proof

Since

$$A_t \subseteq A_s,$$

any relation formed from two atoms in A_t is still formed from the same two atoms in A_s. The appearance of higher-layer atoms changes neither the existing endpoints nor their existing atomic identities.

Thus every old relation remains directly as the same relation instance.

This proves the theorem.

Corollary 3.4 — Preservation of Readout Labels

Part I defined the lossless relation coordinates

$$\Psi(r) = (\sigma(r), \delta(r)).$$

Then

$$\boxed{\Psi(\iota_{t,s} r) = \Psi(r).} \tag{3.4}$$

Once a relation enters a higher layer, its sum label, difference label, and endpoint identities remain unchanged.

Figure 2-03 Relation Birth Layer and Persistent Inheritance

A relation is born at the layer where its later endpoint first appears; once confirmed, every later layer preserves the same relation and readout label.

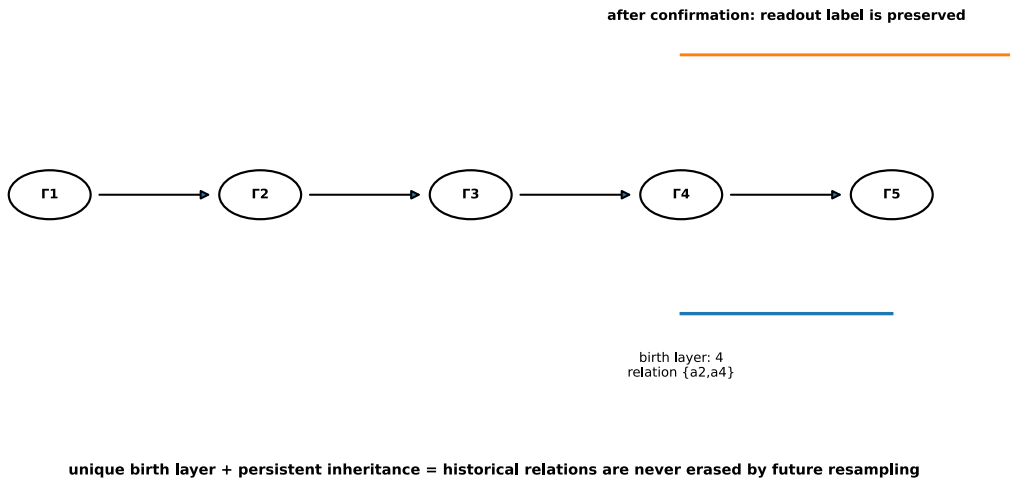


Figure 2-03. Relation Birth Layer and Persistent Inheritance. A relation is born at the unique layer where its later endpoint first enters the matrix. Once confirmed, it persists through every higher layer together with its existing readout label, joining one-time birth and permanent preservation into a single history.

Chapter 4 — Candidate Relations and Confirmed Relations

If candidate relations and confirmed relations are not distinguished, an apparent contradiction arises:

on one hand, some relations in a relation sieve are “killed”;

on the other hand, confirmed relations are permanently persistent.

The two statements refer to different objects.

Definition 4.1 — Candidate Endpoint

An odd position x, before all required multiplicative-layer certifications have been completed, is called a **candidate atomic position**.

It may ultimately become an atom, or it may be covered by a historical atomic ray.

Definition 4.2 — Candidate Relation Shell

A relational object consisting of finitely many candidate endpoints is called a **candidate relation shell**, denoted

$$C.$$

A candidate relation may be eliminated at a multiplicative layer.

Definition 4.3 — Permanent Relation Skeleton

If every endpoint of a finite relation has been confirmed to be an atom, the relation is promoted to a **permanently confirmed relation** in the matrix.

In the two-atom case, it enters

$$\Gamma_{\infty}.$$

Thus the matrix dynamics is not

$$\text{confirmed relation} \to \text{death},$$

but rather

$$\boxed{\text{candidate relation} \to \begin{cases}\text{elimination},\\ \text{confirmation and promotion.}\end{cases}} \tag{4.1}$$

Once promoted, it is preserved permanently by Theorem 3.3.

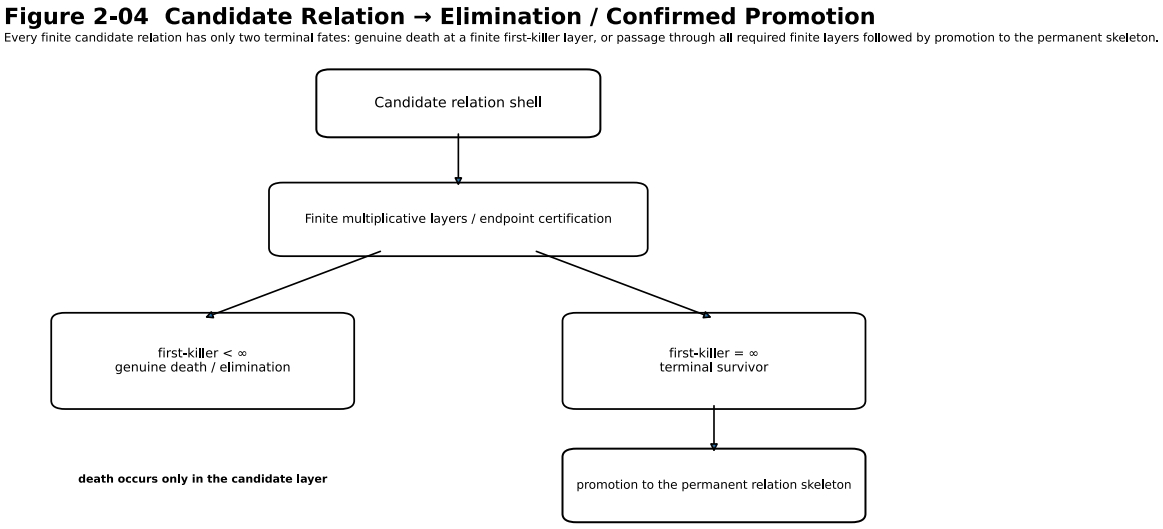


Figure 2-04. Candidate Relation → Elimination / Confirmed Promotion. After a finite candidate relation passes through the required multiplicative certification layers, it has only two terminal fates: if a finite first-killer exists, it genuinely dies at that unique responsibility layer; if all endpoints are atomic, it survives a finite terminal horizon and is irreversibly promoted to a permanent relation. There is no permanently undecided third state.

Chapter 5 — Relation-Faithful States: Why One Count Is Usually Not Enough

The unity of the fixed spectrum does not mean that every relation can be compressed into a scalar.

On the contrary, once a relation involves concrete pairing, position itself is information.

Consider a fixed even integer N.

Define the unordered fixed-sum relation slots

$$\boxed{\mathcal{S}_N = \{(x, N-x) : 3 \le x \le N/2,\ x \text{ odd}\}.} \tag{5.1}$$

If one records only

$$|\mathcal{S}_N|,$$

or only the number of remaining slots at a given sieve layer, one may lose the residue configuration needed for the next update.

5.1 An Explicit Example

Take

$$N_1 = 2006, \qquad N_2 = 2008.$$

They have the same initial number of slots,

$$H = 501.$$

After processing the prime layer

$$q = 3,$$

both have

$$168$$

candidate slots remaining.

Thus if the state stores only

$$(H, |S_3|),$$

the two targets are indistinguishable.

But at the next layer,

$$q = 5,$$

N_1 loses

$$66$$

new slots, while N_2 loses

$$67.$$

The next states are therefore

$$102, \qquad 101.$$

Hence

$$\boxed{(H, |S_q|) \text{ does not determine the next-layer update.}} \tag{5.2}$$

This shows that

the locations and residue configurations of concrete relation slots carry information lost by scalar counts.

(5.3)

Definition 5.2 — Relation-Faithful State

For a finite relation family $\mathcal{C}$, a state is called **relation-faithful** if it preserves at least

$$\boxed{\text{relation-slot positions} + \text{endpoints} + \text{current identities} + \text{necessary death history},} \tag{5.4}$$

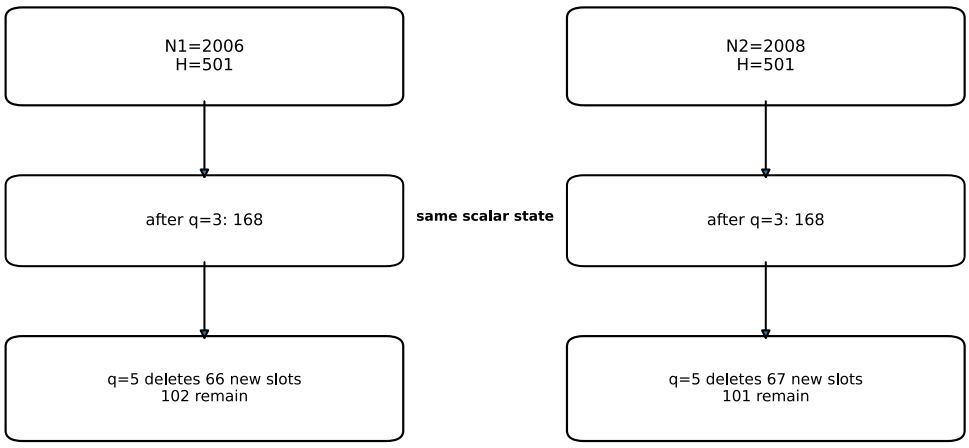
and if this information is sufficient to determine the exact next-layer update.

Relation fidelity does not require that all raw information be preserved forever.

It requires that no compression delete information on which the current proof target actually depends.

Figure 2-05 Why One Count Is Not Enough: 2006 versus 2008

The two fixed-sum targets have identical scalar state after the q=3 layer, yet different residue configurations cause them to diverge at q=5.



(H, |S_q|) does not determine the next update; slot positions and residue configuration are themselves state information.

Figure 2-05. Why One Count Is Not Enough: 2006 versus 2008. In the explicit example from the text, $N_1 = 2006$ and $N_2 = 2008$ both begin with $H = 501$ slots and both leave 168 candidates after $q = 3$. At $q = 5$, however, they newly delete 66 and 67 slots respectively, giving 102 and 101 survivors. Thus $(H, |S_q|)$ does not determine the next update.

Chapter 6 — Unified Moving Horizons: How a Finite Window Receives Zero-Error Atomic Certification

For any real number

$$X \geq 3,$$

define the reduced-coordinate window

$$\boxed{\mathcal{V}(X) = \{a \geq 1 : \Phi(a) = 2a + 1 \leq X\}.} \tag{6.1}$$

This is a finite set.

Let

$$\mathcal{P}_{\sqrt{X}} = \{p \text{ an odd prime} : p \leq \sqrt{X}\}.$$

The corresponding reduced atomic set is

$$A_{\sqrt{X}} = \left\{\frac{p-1}{2} : p \in \mathcal{P}_{\sqrt{X}}\right\}.$$

Define the multiplicative occupancy field generated by historical atoms inside the window by

$$\boxed{\mathcal{M}(X) = \bigcup_{a_i \in A_{\sqrt{X}}} (a_i \star \mathbb{N}_{>0}) \cap \mathcal{V}(X).} \tag{6.2}$$

Theorem 6.1 — Atomic Certification by a Finite Moving Horizon

For every

$$a \in \mathcal{V}(X),$$

we have

$$\boxed{a \text{ is a } \star\text{-atom} \Leftrightarrow a \notin \mathcal{M}(X).} \tag{6.3}$$

Returning to ordinary odd integers,

$$\boxed{2a + 1 \leq X \text{ is prime} \Leftrightarrow 2a + 1 \text{ has no nontrivial prime divisor } p \leq \sqrt{X}.} \tag{6.4}$$

Proof

If

$$a \in \mathcal{M}(X),$$

there exist a historical atom a_i and a positive integer b such that

$$a = a_i \star b.$$

Then

$$\Phi(a) = \Phi(a_i)\Phi(b),$$

with both factors larger than 1, so $\Phi(a)$ is composite.

Conversely, suppose

$$\Phi(a) \leq X$$

is odd and composite. It has a least prime factor q satisfying

$$q \le \sqrt{\Phi(a)} \le \sqrt{X}.$$

Hence

$$q \in \mathcal{P}_{\sqrt{X}}.$$

Write

$$q = \Phi(a_i).$$

The other factor is also an odd integer at least 3, so it can be written as

$$\Phi(b).$$

Thus

$$\Phi(a) = \Phi(a_i)\Phi(b) = \Phi(a_i \star b).$$

Since Φ is injective,

$$a = a_i \star b.$$

Hence

$$a \in \mathcal{M}(X).$$

This proves the theorem.

Corollary 6.2 — Zero-Error Terminal Certification of a Finite Window

Once all atomic layers not exceeding

$$\sqrt{X}$$

have been generated, the atomic identity of every position in the window

$$\mathcal{V}(X)$$

is completely determined.

No larger new atom can later turn an uncovered position inside the window into a composite.

Therefore,

$$\boxed{\mathcal{V}(X) \text{ is a genuine terminal horizon, not an approximate sieve window.}} \tag{6.5}$$

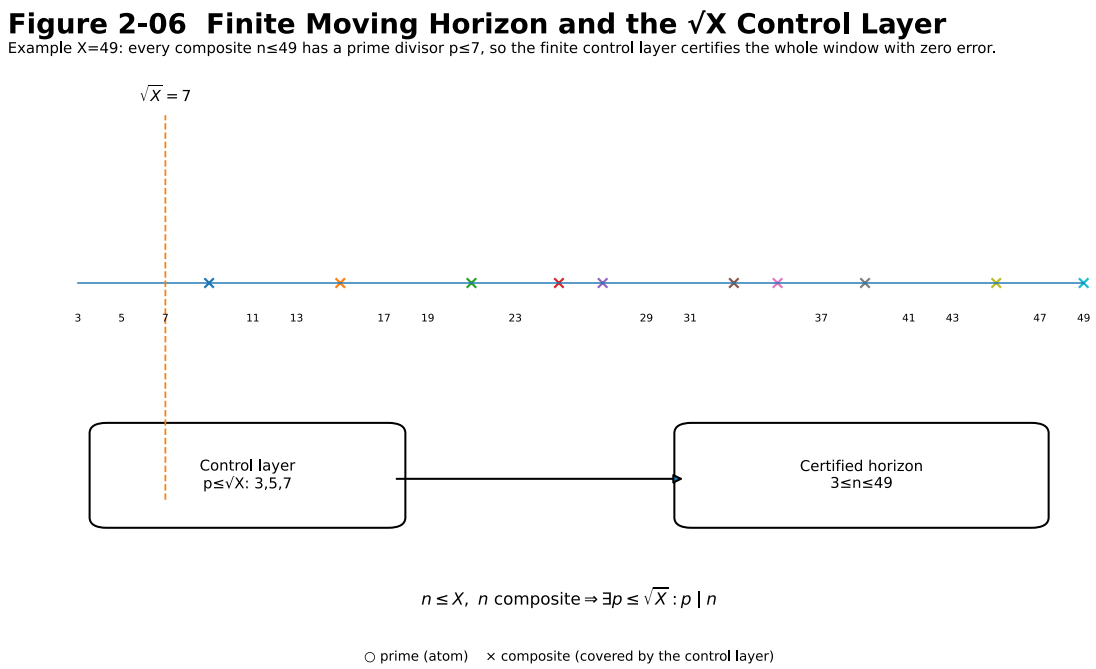


Figure 2-06. Finite Moving Horizon and the √X Control Layer. Within a finite horizon $[1, X]$, every composite integer has a prime divisor not exceeding $\sqrt{X}$. Hence a finite set of historical atoms suffices to certify the entire horizon with zero error.

Chapter 7 — Prime Squares: The Natural Discrete Boundaries of Moving Horizons

The continuous variable X may move arbitrarily, but the historical prime layers needed for atomic certification change only when

$$\sqrt{X}$$

crosses a new prime.

Let

$$p_k < p_{k+1}$$

be consecutive odd primes.

Define the square window

$$\boxed{W_k = [p_k^2, p_{k+1}^2).} \tag{7.1}$$

Theorem 7.1 — Fixed-History Theorem for a Square Window

For every odd integer

$$n \in W_k,$$

if n is composite, then n has at least one prime factor not exceeding p_k.

Thus throughout the whole of W_k, the multiplicative historical layers needed for primality certification are fixed as

$$\{3,5,\dots,p_k\}.$$

Proof

If n is composite, there is a prime factor

$$q \le \sqrt{n}.$$

Since

$$n < p_{k+1}^2,$$

we have

$$q < p_{k+1}.$$

Because p_k, p_{k+1} are consecutive,

$$q \le p_k.$$

This proves the theorem.

Therefore,

$$\boxed{W_1, W_2, W_3, \dots}$$

form the natural discrete layers of the moving horizon.

On passing to the next square window, only one new atomic divisor layer needs to be added.

Thus

Prime squares are not specially imposed auxiliary boundaries;
they are the natural locations at which the historical layers required for finite atomic certification change.

(7.2)

Moreover:

How far a horizon must move is determined by the layer of historical information sufficient to render the finite relation under study permanently terminal.

Different relations may possess more economical adapted terminal boundaries, but all obey the same finite-terminal principle.

Figure 2-07 Natural Discrete Horizons from Consecutive Prime Squares

Within W_k=[p_k²,p_{k+1}²), the historical prime layers needed to certify composites are fixed: all primes up to p_k.

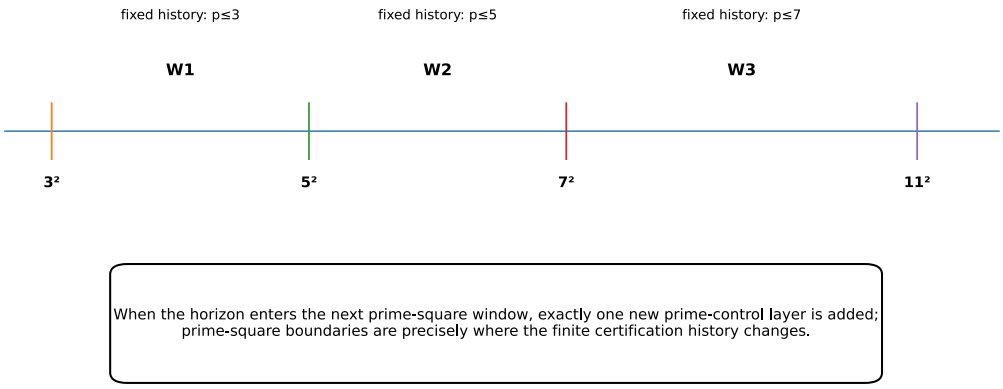


Figure 2-07. Natural Discrete Horizons from Consecutive Prime Squares. For consecutive odd primes $p_k < p_{k+1}$, every composite in the square window $W_k = [p_k^2, p_{k+1}^2)$ has a prime divisor at most p_k. Hence the historical control layers are fixed throughout the window; one new control layer is added only when the horizon crosses the next prime square.

Chapter 8 — Genuine Death Layers: A Composite Dies for the First Time Only Once

An ordinary divisibility sieve can hit the same integer through several prime factors.

To build a nonoverlapping historical decomposition, one must specify the first layer at which each composite becomes certified composite.

For an odd composite n, define

$$\boxed{\kappa(n) = P^{-}(n).} \tag{8.1}$$

Definition 8.1 — Genuine Death Layer

For an odd prime q, define

$$\boxed{D_q = \{n \text{ odd composite}: P^{-}(n) = q\}.} \tag{8.2}$$

Then

$$\boxed{D_q \cap D_r = \varnothing \qquad (q \neq r).} \tag{8.3}$$

Every odd composite belongs to exactly one D_q.

Therefore,

Every odd composite undergoes genuine death exactly once, at the layer of its least prime factor.

(8.4)

Theorem 8.2 — Completeness of Genuine Death in a Finite Window

Within the finite window

$$n \leq X,$$

every odd composite belongs to some

$$D_q, \qquad q \leq \sqrt{X}.$$

Thus after all genuine-death layers

$$q \leq \sqrt{X}$$

have been executed, the remaining positions are exactly the odd primes in the window.

Proof

This follows directly from the least-prime-factor property.

This theorem and Theorem 6.1 are two formulations of the same finite-terminal fact:

- the language of multiplicative occupancy emphasizes which positions are covered by historical rays;
- the language of genuine death emphasizes the first layer at which each composite position exits.

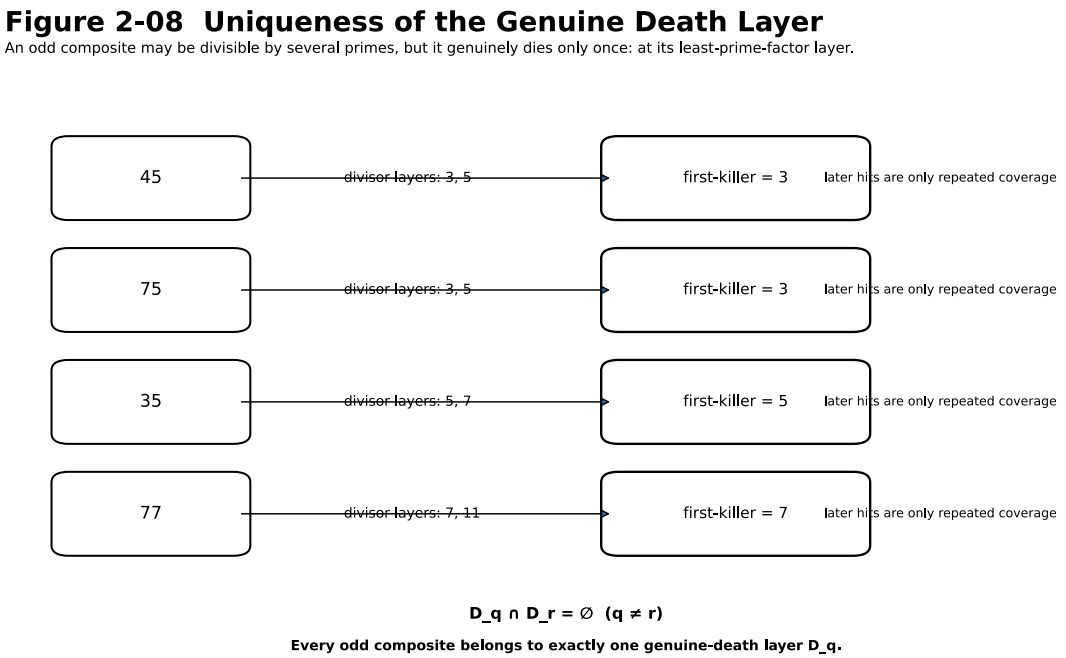


Figure 2-08. Uniqueness of the Genuine Death Layer. The genuine-death layer of an odd composite n is uniquely determined by $\kappa(n) = P^{-}(n)$. Although larger prime divisors may hit the same composite again, those hits are only repeated coverage; genuine death occurs only at the least-prime-factor layer, so the sets D_q are pairwise disjoint.

Chapter 9 — Relation-Faithful Sieves: From Endpoint Death to Relation Death

Genuine death can be lifted from single integers to general finite relations.

Let a relation instance be

$$r = (n_1, \dots, n_m).$$

If the local predicate requires every endpoint to be prime, then any composite endpoint is sufficient to make the relation fail.

Definition 9.1 — Relation First Killer

If at least one endpoint is composite, define

$$\boxed{\kappa(r) = \min\{P^-(n_i) : n_i \text{ is composite}\}.} \tag{9.1}$$

Call $\kappa(r)$ the relation's **first responsible prime**, or **first killer**.

If every endpoint is prime, define

$$\kappa(r) = \infty.$$

Theorem 9.2 — Unique Genuine Death of a Relation

Every finite all-prime-type candidate relation r has exactly one of the following two forms:

$$\boxed{\begin{cases} \kappa(r) < \infty, & r \text{ undergoes its first genuine death at the unique first-killer layer;} \\ \kappa(r) = \infty, & \text{all endpoints of } r \text{ are prime.} \end{cases}} \tag{9.2}$$

Proof

If there are composite endpoints, the finite set

$$\{P^-(n_i) : n_i \text{ is composite}\}$$

has a unique minimum.

At every smaller prime layer, no endpoint has yet been certified dead; at the $\kappa(r)$-layer, at least one endpoint is certified composite for the first time.

If there is no composite endpoint, all endpoints are prime.

This proves the theorem.

9.1 Fixed-Sum Relation Slots

For a fixed even integer N, a relation slot

$$(x, N - x)$$

undergoes genuine death only if

$$q \mid x, \quad x > q,$$

or

$$q \mid N - x, \quad N - x > q.$$

The prime q itself is not deleted merely because

$$q \mid q.$$

This is the fundamental distinction between a relation-faithful sieve and the simple instruction "delete every multiple of q."

9.2 Terminal Exactness

If all candidate endpoints for a fixed sum are at most $N-3$, then after processing

$$q \le \sqrt{N-3},$$

the terminal surviving slots are exactly the genuine prime–prime slots:

$$\boxed{\text{surviving slots of the terminal relation sieve} = \text{genuine all-atomic relation slots.}} \tag{9.3}$$

The function of a relation sieve is therefore not approximation but exact certification inside a finite horizon.

Chapter 10 — Terminal Certification of Finite Relations

Definition 10.1 — Finitely Supported Relation

Let

$$R = (x_1, \dots, x_r)$$

be a finite relational object, where each x_j is a reduced odd coordinate.

Its ordinary odd endpoints are

$$\Phi(x_1), \dots, \Phi(x_r).$$

If all endpoints are atoms, call R a **genuine atomic relation**.

Define

$$\boxed{M(R) = \max_{1 \le j \le r} \Phi(x_j).} \tag{10.1}$$

Theorem 10.2 — Terminal Certification of a Finite Relation

The identities of all endpoints of any finitely supported relation R can be certified with zero error inside the finite horizon

$$\boxed{\mathcal{V}(M(R)).}$$

More concretely, it is sufficient to execute all genuine-death layers

$$q \le \sqrt{M(R)}.$$

Proof

For every endpoint,

$$\Phi(x_j) \le M(R),$$

so by Theorem 6.1 its primality requires checking only prime-factor layers not exceeding

$$\sqrt{M(R)}.$$

Because there are finitely many endpoints, they all complete certification simultaneously inside the same finite horizon.

This proves the theorem.

Corollary 10.3 — Every Finite Relation Has a Finite Terminal Horizon

Define

$$\tau(R)$$

as the least horizon layer at which the identities of all endpoints of R become permanently determined.

Then

$$\boxed{\tau(R) < \infty.} \tag{10.2}$$

Therefore,

No finite-endpoint prime relation requires infinite information to acquire its final identity.

(10.3)

Theorem 10.4 — Dichotomy for Candidate Relations

Every finite candidate relation R lies in exactly one of the following two classes:

$$\begin{cases} \kappa(R) < \infty, & \text{it is genuinely eliminated at a finite multiplicative layer;} \\ \kappa(R) = \infty, & \text{all endpoints are atoms and terminal certification is completed in a finite horizon.} \end{cases} \tag{10.4}$$

There is no third state of permanent indecision.

Proof

The number of endpoints is finite.

If at least one endpoint is composite, its least prime factor is finite, so

$$\kappa(R) < \infty.$$

If there is no composite endpoint, all endpoints are prime; by Theorem 10.2, they are all certified inside a finite horizon.

This proves the theorem.

Chapter 11 — Promotion from Candidate Shells to the Permanent Relation Skeleton

Define the promotion map

$$\boxed{\Pi: \mathcal{C}^{\mathrm{surv}} \to \Gamma^{\mathrm{perm}},} \tag{11.1}$$

where

- $\mathcal{C}^{\mathrm{surv}}$ denotes candidate relations that have passed all necessary finite multiplicative layers;
- Γ^{perm} denotes the permanently confirmed relation skeleton.

In the two-atom case,

$$\Gamma^{\mathrm{perm}} = \Gamma_{\infty}.$$

Theorem 11.1 — Irreversibility of Promotion

If a candidate relation R has completed terminal certification and has been promoted to a permanent atomic relation, then it remains genuine in every larger horizon.

That is,

$$\boxed{R \in \Gamma_t \Rightarrow R \in \Gamma_s \qquad (s \geq t).} \tag{11.2}$$

Proof

Promotion means that all endpoints have already been confirmed as atoms. Prime identity cannot be changed by the later appearance of larger primes.

By persistent embedding,

$$\Gamma_t \subseteq \Gamma_s.$$

This proves the theorem.

Therefore,

$$\boxed{\begin{array}{c}\text{candidate relation is born}\\ \downarrow\\ \text{finite multiplicative-layer certification}\\ \downarrow\\ \left\{\begin{array}{l}\text{genuine death,}\\ \text{terminal survival}\end{array}\right.\\ \downarrow\\ \text{terminal survivors are promoted}\\ \downarrow\\ \text{permanent relation skeleton.}\end{array}} \tag{11.3}$$

“Death” occurs only at the candidate layer;

“history is never lost” applies only to the already promoted permanent layer.

The two are fully compatible.

Chapter 12 — Relation Histories: An Object Has Not Only a Result but a Generative History

Definition 12.1 — Relation History

For a finite relation R, define

$$\boxed{\mathcal{H}(R) = (\mathrm{supp}R,\ b(R),\ \tau(R),\ \kappa(R),\ \mathsf{Lab}(R),\ \mathsf{Status}(R)).} \tag{12.1}$$

where:

- $\mathrm{supp}R$: the finite endpoint set;
- $b(R)$: the birth layer at which the relation enters the candidate or confirmed structure;
- $\tau(R)$: the least terminal horizon that makes its identity permanently determined;
- $\kappa(R)$: the genuine death layer; for a genuine atomic relation, $\kappa(R) = \infty$;
- $\mathsf{Lab}(R)$: sum, difference, pattern, window, or other readout labels;
- $\mathsf{Status}(R)$: candidate, dead, or permanently confirmed status.

Theorem 12.2 — History-Preservation Theorem

If R has already been promoted to a permanent relation, then every higher horizon preserves

$$\mathrm{supp}R, \qquad b(R), \qquad \mathsf{Lab}(R), \qquad \mathsf{Status}(R) = \mathrm{persistent}.$$

Higher layers add new relation histories; they do not rewrite old histories.

Thus the matrix is not a relation table recomputed from scratch at every layer, but an ever-growing historical ledger:

$$\boxed{\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots.} \tag{12.2}$$

This is the lowest-order rigorous form of the statement “a relation has a history.”

Figure 2-09 Relation-History Ledger

A relation object records not only its final truth value, but also support, birth, terminality, death responsibility, readout label, and current status.

Field	Meaning	Typical value
supp R	finite endpoint set	{n1,...,nm}
b(R)	relation birth layer	k
τ(R)	minimal terminal horizon	finite
κ(R)	genuine-death layer	q or ∞
Lab(R)	sum / gap / template / window label	σ,δ,H,I_n,...
Status(R)	candidate / dead / persistent	candidate / dead / persistent

Higher horizons append new histories; they do not rewrite previously confirmed persistent histories.

L1 ⊆ L2 ⊆ L3 ⊆ ⋯

Figure 2-09. Relation-History Ledger. The relation history $\mathcal{H}(R)$ records the finite support $\mathrm{supp}R$, birth layer $b(R)$, terminal horizon $\tau(R)$, genuine-death layer $\kappa(R)$, readout label $\mathsf{Lab}(R)$, and state $\mathsf{Status}(R)$. Higher layers append new histories rather than rewriting histories already confirmed as permanent.

Chapter 13 — Birth and Death Are Strictly Misaligned in Scale

Once a new prime

$$q$$

is born as an atom, it immediately forms new additive relations with all historical atoms.

But if q is viewed as a new least-prime-factor death layer, the first genuinely new composite death for which it is responsible cannot occur before

$$q^2.$$

Indeed, if

$$n = qm$$

and

$$m < q,$$

then m contains some prime factor

$$r < q,$$

so n already died at the earlier r-layer.

Therefore,

$$\boxed{\text{the first integer whose least prime factor is } q \text{ is } q^2.} \tag{13.1}$$

Theorem 13.1 — Time-Offset Theorem Between Atomic Birth and Genuine Death

The two roles of a new atom q satisfy

relational channel:	q creates new relations immediately at birth;
death channel:	q begins to create genuine deaths as a new least prime factor only from q^2

(13.2)

Hence

irreducible birth occurs first;

new least-factor death occurs later.

(13.3)

An atom does not passively wait for relations to appear. At birth it contributes an entire layer of structure to the relation matrix, whereas its exclusive multiplicative role in genuine death becomes visible only at a larger scale.

Chapter 14 — A Moving Horizon Does Not Create Relations; It Reads Them Terminally

Genuine relations already enter the historical matrix at atomic birth.

The role of a moving horizon is not to regenerate these relations, but to answer:

> Which finite relations have entered a terminal region that the future can no longer rewrite?

Therefore,

$$\boxed{\text{relation birth} \neq \text{entry of the relation into a terminal horizon.}} \tag{14.1}$$

A relation may have existed in the historical matrix for some time, yet only at a later horizon does it receive terminal certification as a complete finite object for a particular readout.

Thus

> **A moving horizon is not a relation generator, but a device for terminalizing and reading relations.**

(14.2)

This is especially relevant to fixed sums and fixed gaps:

- a fixed-sum relation may already have been born, while a particular target fiber is only completely read at its adapted terminal horizon;
- a finite fixed-gap candidate can complete two-endpoint certification inside a square horizon and be promoted to a permanent genuine relation.

Chapter 15 — Unified Horizons and Readout-Adapted Horizons

The entire book uses one moving-horizon principle:

Once all historical atomic layers capable of changing the identity of a finite relation have entered, that relation becomes terminally frozen.

(15.1)

The general form is

$$\mathcal{V}(X),$$

with certification depth

$$\sqrt{X}.$$

Different relations, however, may derive more economical **readout-adapted terminal horizons** from the same principle.

This is not a second theory; it is a minimization of the same finite-terminal principle.

15.1 Adapted Terminal Horizon for a Fixed-Sum Readout

If a two-atom relation satisfies

$$a_i + a_j + 1 = m,$$

then each endpoint is strictly smaller than m.

Therefore, once the historical atoms have passed all possible endpoints that can participate in this sum, no larger atom can alter the fixed-sum fiber.

A fixed sum thus has its own finite natural terminal segment.

15.2 Adapted Terminal Horizon for Fixed Finite Endpoints

If the relation endpoints are explicitly given—for example, a fixed-gap endpoint pair or a finite pattern—it is sufficient to proceed to square-root depth relative to the largest endpoint.

Thus fixed gaps and general finite patterns naturally obey the square-terminal principle.

15.3 Adapted Terminal Horizon for a Window Readout

If the target asks only whether a finite window

$$I = [L, U]$$

contains a prime, then the certification depth is determined by the largest endpoint U.

After all genuine-death layers

$$q \le \sqrt{U}$$

have been processed, the prime set inside the window is completely determined.

15.4 Adapted Terminal Horizon for a General Finite Relation

For a family of position maps

$$f_1, \dots, f_m,$$

at a fixed parameter instance (Θ, x), define

$$M_{\Theta,x} = \max_i |f_i(x;\Theta)|.$$

As long as all target positions are positive integers and the local predicate depends only on the finite arithmetic identities of those positions, the bound

$$\boxed{\tau(\Theta, x) \le \sqrt{M_{\Theta,x}}} \tag{15.2}$$

provides a unified terminal upper bound.

Therefore,

A terminal horizon is precisely a horizon that already contains all historical information capable of changing the identity of the current finite relation.

(15.3)

Chapter 16 — Unified Normal Form for Relation Reconstruction

Part I gave the local relation normal form

$$R_\Theta(x) = \mathcal{B}(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_m(x;\Theta))).$$

This part augments it with relation history.

Define the complete finite relation state

$$\boxed{\mathfrak{R}_{\Theta,x} = (\Theta, x, (f_i)_{i=1}^m, (f_i(x;\Theta))_{i=1}^m, (\boldsymbol{I}(f_i(x;\Theta)))_{i=1}^m, \mathcal{B}, \mathcal{H}).} \tag{16.1}$$

where:

- the parameter Θ determines the relation family;
- the anchor x determines the concrete relation instance;
- f_i preserve the position maps;
- $f_i(x;\Theta)$ preserve the concrete positions;
- $\boldsymbol{I}$ preserves fixed identities;
- $\mathcal{B}$ gives the local relation predicate;
- $\mathcal{H}$ preserves the history of relation birth, death, terminalization, and persistence.

This is the complete finite object for relation reconstruction.

If some compression still allows all target-relevant structure to be recovered, that compression may be used; if not, the compression must not replace the relational object.

Chapter 17 — How Finite Horizons Enter Infinite Quantifiers

Finite terminality explains how each finite relation acquires a final identity.

When an infinite quantifier enters, the question becomes:

> *How do these finite terminal results cover an entire infinite parameter family?*

This does not follow automatically from the fact that each instance is finite.

The outer quantifier must be treated according to its type.

17.1 Infinite Recurrence

Let

$$R(x) \in \{0,1\}.$$

Define the cumulative count

$$\boxed{C_R(X) = \sum_{x \le X} R\,(x).} \tag{17.1}$$

Then the exact equivalence is

$$\boxed{\exists^{\infty} x: R(x) = 1 \Leftrightarrow C_R(X) \text{ is unbounded.}} \tag{17.2}$$

Proof

If infinitely many hits occur, the cumulative count exceeds every prescribed integer.

Conversely, if the cumulative count is unbounded, the set of hits cannot be finite.

This proves the equivalence.

This quantifier type applies to Twin, infinite recurrence of a fixed pattern, infinitude of Mersenne primes, and similar problems.

17.2 Parameterwise Universal Fibers

Let

$$R_\theta(x)$$

be a local relation at parameter θ.

Define the fiber count

$$\boxed{C(\theta) = \sum_{x} R_\theta\,(x).} \tag{17.3}$$

Then

$$\boxed{\forall \theta\ \exists x: R_\theta(x) = 1 \Leftrightarrow C(\theta) > 0 \quad \forall \theta.} \tag{17.4}$$

This quantifier type applies to fixed-sum problems.

Thus the central task of a universal proof is not to search manually for a witness one target at a time, but to establish a uniform positive lower bound

$$\boxed{C(\theta) \ge L(\theta) > 0.} \tag{17.5}$$

17.3 Window Universality

Given a window family

$$I_n,$$

define

$$\boxed{W(n) = \sum_{m \in I_n} \alpha\,(m).} \tag{17.6}$$

Then

$$\boxed{\forall n\ \exists p \in I_n \cap \mathbb{P} \Leftrightarrow W(n) \geq 1 \quad \forall n.} \tag{17.7}$$

Legendre and Oppermann are of this type.

17.4 Arbitrary-Length Relations

Let

$$R_{k,\theta}(x)$$

denote a relation with length parameter k.

Define the largest realized length up to horizon X by

$$\boxed{L(X) = \max\{k: \exists \theta, x \leq X,\ R_{k,\theta}(x) = 1\}.} \tag{17.8}$$

Then

$$\boxed{\forall k\ \exists \theta, x: R_{k,\theta}(x) = 1 \Leftrightarrow L(X) \text{ is unbounded.}} \tag{17.9}$$

This quantifier type applies to arbitrarily long arithmetic progressions of primes.

17.5 Infinitely Many Parameters

Sometimes the issue is not infinitely many anchor hits but infinitely many successful parameter values.

Define

$$\boxed{P(X) = \#\{\theta \leq X: \exists x,\ R_\theta(x) = 1\}.} \tag{17.10}$$

Then

$$\boxed{\exists^\infty \theta\ \exists x: R_\theta(x) = 1 \Leftrightarrow P(X) \text{ is unbounded.}} \tag{17.11}$$

17.6 Indexwise Universal Predicates

Problems such as Andrica, Firoozbakht, Brocard, and Grimm have the form

$$\boxed{\forall n, \quad P(n).} \tag{17.12}$$

Define the failure count

$$\boxed{F(X) = \sum_{n \leq X} \mathbf{1}_{\{\neg P(n)\}}.} \tag{17.13}$$

The universal proposition is equivalent to

$$\boxed{F(X) = 0 \quad \text{for every } X.} \tag{17.14}$$

If one can establish a uniform mechanism showing that a least failure point cannot exist, the universal proposition closes.

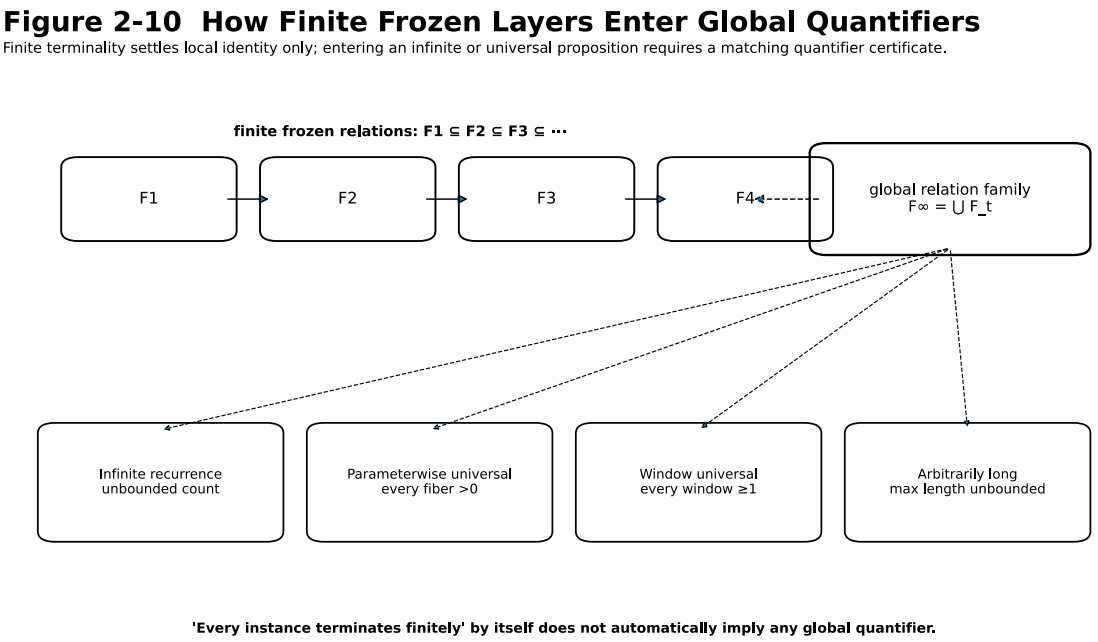


Figure 2-10. How Finite Frozen Layers Enter Global Quantifiers. Finite terminality says only that each finite object eventually freezes; a global proposition still requires a certificate matched to its outer quantifier. Infinite recurrence reads unbounded cumulative count, parameterwise universality reads nonempty fibers, window universality reads a hit in every window, and arbitrary length reads an unbounded maximum realized length.

Chapter 18 — Finite-to-Infinite Entry Theorem

Suppose a class of relations forms a frozen set

$$F_t$$

at finite terminal layer t, and assume

$$F_t \subseteq F_{t+1}.$$

Define the global relation family

$$\boxed{F_\infty = \lim_{\rightarrow t} F_t = \bigcup_{t \geq 1} F_t.} \tag{18.1}$$

Theorem 18.1 — Finite-to-Infinite Entry Theorem

Under persistence,

$$\boxed{F_\infty \text{ is infinite} \Leftrightarrow \forall T,\ \exists t > T \text{ such that } F_t \backslash F_T \neq \varnothing.} \tag{18.2}$$

Equivalently,

$$\boxed{\text{the global relation family is infinite} \Leftrightarrow \text{there is no last moving horizon that produces a new froz}} \tag{18.3}$$

Proof

If F_∞ is infinite but there exists T such that for every $t > T$,

$$F_t = F_T,$$

then

$$F_\infty = F_T.$$

But F_T is a finite relation set in a finite horizon, a contradiction.

Conversely, if every T is followed by a later layer that produces at least one new frozen relation, one can recursively choose a strictly increasing sequence of relations

$$R_1, R_2, \ldots,$$

with each R_j first appearing in a later horizon. Hence F_∞ contains infinitely many distinct relations.

This proves the theorem.

Therefore,

Infinity is not the one-time verification of a completed infinite object;
it is the continuing birth of finite frozen relations with no final stopping layer.

(18.4)

Chapter 19 — How Different Global Quantifiers Enter the Same Matrix

Fixed-sum problems require

$$\forall\theta\ \exists x.$$

Infinite-recurrence problems require

$$\exists^{\infty}x.$$

Window problems require

$$\forall n\ \exists x \in I_n.$$

Arbitrary-length problems require

$$\forall k\ \exists \theta, x.$$

These quantifiers differ, yet they share the same finite foundation:

1. every finite relation instance has a finite terminal horizon;
2. every frozen relation remains permanently fixed;
3. new atoms continue to produce new candidate and newly confirmed relations;
4. every global proposition is built on finite terminal relations.

Therefore,

$$\boxed{\text{a unified matrix} \neq \text{a unified quantifier.}} \tag{19.1}$$

What is unified is the relational object and finite terminal structure;

different quantifiers must use their own correct global criterion.

This is also why **batch proof** becomes possible:

First unify the local relations;

then treat the whole parameter family uniformly according to quantifier type.

(19.2)

Chapter 20 — The Finite Foundation of Batch Proof

Checking relation instances one by one yields only finite facts.

To cover a parameter family, a unified certificate is needed.

Let the parameter space be

$$\Theta,$$

with target proposition

$$\mathcal{P}(\theta).$$

Definition 20.1 — Batch Certificate

Let

$$\mathcal{S}$$

be a certificate-state space.

If there exists a map

$$\boxed{\mathcal{C}:\Theta \to \mathcal{S}} \tag{20.1}$$

satisfying:

Coverage

every

$$\theta \in \Theta$$

enters a valid certificate state;

Correctness

every valid certificate state implies

$$\mathcal{P}(\theta),$$

then $\mathcal{C}$ is called a **batch certificate** for the parameter family.

It follows that

$$\boxed{\forall \theta \in \Theta, \quad \mathcal{P}(\theta).} \tag{20.2}$$

The essence of batch proof is not "computing many instances simultaneously," but rather

> **one finite rule covers an entire parameter class at once.**

(20.3)

Chapter 21 — Four Basic Forms of Finite Closure

For finite structures to enter universal or infinite conclusions, an explicit closure mechanism is required.

21.1 Finite-Quotient Closure

If an infinite parameter family has only finitely many states under an equivalence relation,

$$\Theta/\sim = \{s_1, \dots, s_m\},$$

and all parameters in the same state have the same target conclusion, it suffices to prove the finitely many states.

Typical sources include finite residue classes, CRT states, and finite automaton states.

21.2 Propagation Closure

If there exists a recurrence rule

$$\boxed{P_n \Rightarrow P_{T(n)}} \tag{21.1}$$

and the initial set together with propagation orbits covers all parameters, then finitely many starting cases plus the propagation rule close the universal statement.

21.3 Cumulative-Potential Closure

If a cumulative relation quantity satisfies

$$C(X) \geq E(X),$$

and

$$\boxed{\limsup_{X\to\infty} E(X) = +\infty,} \tag{21.2}$$

then the cumulative number of hits is unbounded, and the relation recurs infinitely often.

21.4 Bad-State Exclusion Closure

If the set of all bad states capable of producing failure is

$$\mathcal{F}_{\text{bad}},$$

and the actually reachable state set

$$\mathcal{R}$$

satisfies

$$\boxed{\mathcal{F}_{\text{bad}} \cap \mathcal{R} = \varnothing,} \tag{21.3}$$

then failure cannot occur.

These closure forms will appear in different concrete relations in different ways.

Chapter 22 — Four Basic Examples

22.1 Fixed Sum

Given an even integer N,

$$R_N(p) = \alpha(p)\alpha(N-p).$$

The local relation for each concrete p terminates inside a finite horizon.

But the universal proposition requires

$$\boxed{\forall N \in 2\mathbb{Z}_{\geq 4}, \quad \sum_p R_N(p) > 0.} \tag{22.1}$$

Thus local terminal certification is not the same as universal coverage.

The key quantity is whether the entire fixed-sum fiber is nonempty.

22.2 Fixed Gap

Given d,

$$R_d(p) = \alpha(p)\alpha(p+d).$$

Each candidate pair completes terminal certification within

$$\sqrt{p+d}.$$

If the problem asks for infinitely many such pairs, one should establish

$$\boxed{\sum_{p \leq X} R_d(p) \to \infty} \tag{22.2}$$

or an equivalent unboundedness statement.

22.3 Square Window

Given

$$I_n = (n^2, (n+1)^2),$$

terminal certification of the window is finite.

If the proposition requires every window to be nonempty, one must establish

$$\boxed{\sum_{m \in I_n} \alpha(m) \geq 1 \qquad \forall n.} \tag{22.3}$$

22.4 Finite Pattern

Given

$$H = \{h_1, \dots, h_k\},$$

define

$$R_H(n) = \prod_{j=1}^{k} \alpha\,(n + h_j).$$

For fixed n, the pattern instance is completely terminal inside a finite horizon.

But "the pattern occurs infinitely often" requires an unbounded cumulative count:

$$\boxed{C_H(X) = \sum_{n \le X} R_H\,(n) \to \infty \text{ or at least is unbounded.}} \tag{22.4}$$

Thus the quantifier layer always lies between finite terminality and an infinite relation.

Chapter 23 — Unified Dynamics: Birth, Elimination, Promotion, Persistence, and Readout

Combining Parts I and II gives the full dynamics.

On the multiplicative side:

$$\boxed{\text{historical multiplicative occupancy} \to \text{least escape} \to \text{birth of a new atom.}}$$

On the relational side:

$$\boxed{\text{new atom} \to \text{an entire new relation layer.}}$$

At the candidate layer:

$$\boxed{\text{candidate relation} \to \begin{cases}\text{genuine death at the first-killer layer,}\\ \text{survival through all necessary finite layers.}\end{cases}}$$

For terminal survivors:

$$\boxed{\text{terminal certification} \to \text{promotion to a permanent relation} \to \text{historical nondeletion.}}$$

At the readout layer:

$$\boxed{\text{moving horizon} \to \text{read terminal relations} \to \text{form a local slice.}}$$

At the quantifier layer:

$$\boxed{\text{local slices} \to \text{batch certificate} \to \text{universal or infinite conclusion.}}$$

Hence the unified dynamics of the book can be written as

$$\boxed{\begin{array}{c}\text{atomic birth}\\ \downarrow\\ \text{relation birth}\\ \downarrow\\ \text{finite candidate certification}\\ \downarrow\\ \text{death or promotion}\\ \downarrow\\ \text{permanent relation skeleton}\\ \downarrow\\ \text{terminal readout by moving horizon}\\ \downarrow\\ \text{relation slice}\\ \downarrow\\ \text{quantifier closure.}\end{array}} \tag{23.1}$$

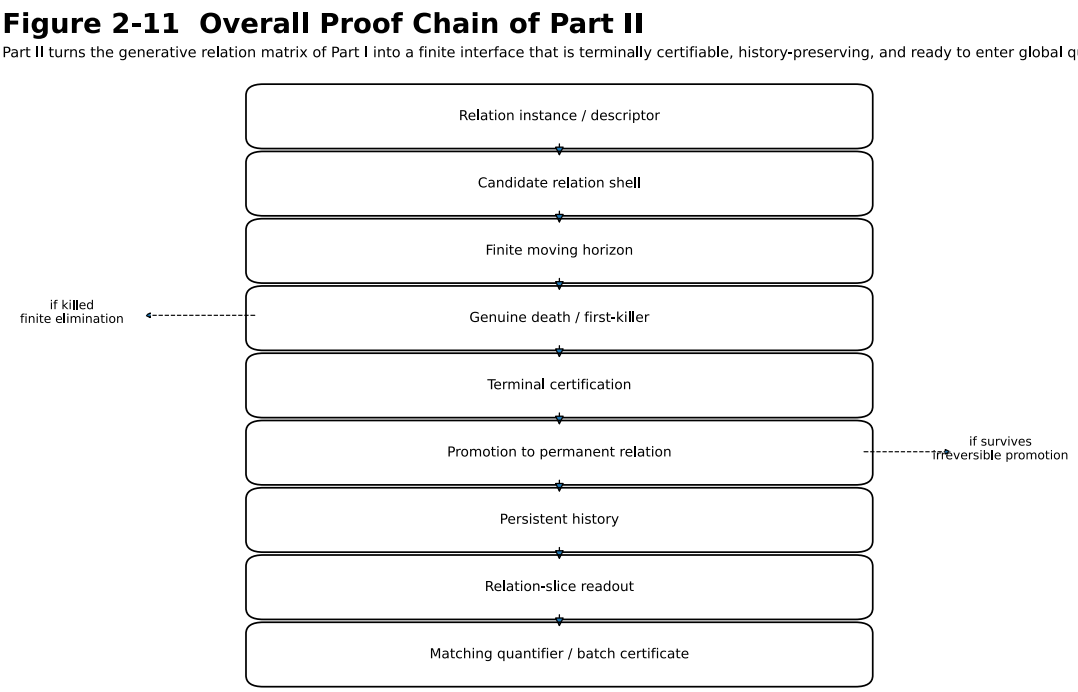


Figure 2-11. Overall Proof Chain of Part II. Part II carries a relation instance through the candidate shell, finite moving horizons, and genuine-death responsibility into terminal certification. Terminal survivors are irreversibly promoted to permanent relations and keep their histories; relation-slice readout plus a matching quantifier then enters the batch certificate. This is the complete interface from finite relations to global propositions.

Chapter 24 — Summary Table of Foundational Theorems in This Part

Foundational Theorem F — Persistent Inheritance of Lower-Order Relations

$$\boxed{\Gamma_t \hookrightarrow \Gamma_s \qquad (s \geq t).}$$

Confirmed relations are only added, never deleted.

Foundational Theorem G — Atomic Certification by a Finite Moving Horizon

For any finite X,

$$\boxed{\Phi(a) \leq X \Rightarrow \text{its atomic identity is completely determined by atomic layers not exceeding } \sqrt{X}.}$$

Foundational Theorem H — Fixed History in a Square Horizon

Inside

$$[p_k^2, p_{k+1}^2),$$

every composite identity is completely decidable from historical prime layers not exceeding p_k.

Foundational Theorem I — Unique Genuine Death

Every odd composite undergoes genuine death exactly once, at the layer of its least prime factor.

Any finite candidate relation that is not an all-atomic relation likewise has a unique first-killer layer.

Foundational Theorem J — Terminality of Finite Relations

Every finitely supported relation R has

$$\boxed{\tau(R) < \infty.}$$

Beyond that point, higher-layer information can no longer alter its local identity.

Foundational Theorem K — Irreversibility of Promotion

Once a finite relation completes atomic certification and enters the permanent relation skeleton, it remains unchanged in all higher horizons.

Foundational Theorem L — Finite-to-Infinite Entry

If a class of frozen relations satisfies persistent embedding, then

$$\boxed{\text{its global relation family is infinite} \Leftrightarrow \text{there is no last moving horizon that produces a new frozen relation.}}$$

Foundational Theorem M — Counting Criterion for Infinite Recurrence

$$\boxed{\exists^{\infty} x: R(x) = 1 \Leftrightarrow \sum_{x \leq X} R\,(x) \text{ is unbounded.}}$$

Foundational Theorem N — Fiber Criterion for Parameterwise Universality

$$\boxed{\forall \theta\ \exists x: R_{\theta}(x) = 1 \Leftrightarrow \sum_{x} R_{\theta}\,(x) > 0 \quad \forall \theta.}$$

Chapter 25 — The Relation Layer Cannot Be Erased by the Fixed Spectrum

The entire structure of this part can be compressed into one important conclusion.

The fixed spectrum provides

$$\boldsymbol{I}(n).$$

But a relational proposition requires

$$\boxed{\text{position maps} + \text{endpoint correspondence} + \text{local predicate} + \text{relation history} + \text{outer quantifier}} \tag{25.1}$$

Therefore,

> the fixed spectrum does not replace relations;
>
> the fixed spectrum makes relations exactly readable.

(25.2)

Nor does the fact that identity is fixed diminish the value of relation theory.

On the contrary, only when the relation layer has been reconstructed to sufficiently low order can apparently different classical problems be placed in the same matrix.

Thus genuine unification is not

“all conjectures are just table lookups.”

It is

> **all conjectures may share one factual spectrum,**
>
> **but every relation must faithfully preserve its own structure.**

(25.3)

This is also the prerequisite for batch proof.

If a relation has already lost information decisive for the target proposition when it enters the unified framework, then no unified language, however powerful, can recover structure that has already been discarded.

Chapter 26 — Discipline Between Finite Layers and Global Propositions

A common misunderstanding is:

> Since every finite relation terminates in a finite horizon, the global conjecture follows automatically.

That is false.

Finite terminality says only

> the truth value of each concrete instance is finitely decidable.

(26.1)

A universal or infinite proposition additionally requires

> the entire parameter family to satisfy a uniform global property.

(26.2)

For example,

$$\forall N\ \exists p$$

does not follow merely from "each given N can be checked finitely."

Likewise,

$$\exists^{\infty} p$$

does not follow merely from "each given p can be decided finitely."

Therefore,

> **finite terminality resolves local decidability;**
> **batch closure resolves global quantifiers.**

(26.3)

Both are indispensable.

Chapter 27 — General Technical Interface for Relation-Faithful Sieves

Relation-faithful sieves are not limited to fixed sums.

For any finite family of positions

$$f_1(x;\Theta),\dots,f_m(x;\Theta),$$

one may construct a state

$$\boxed{\mathcal{S}_{\Theta,x}=((f_i(x;\Theta))_{i=1}^{m},(\mathsf{status}_i)_{i=1}^{m},(\kappa_i)_{i=1}^{m},\kappa(r),\mathsf{Status}(r)).} \tag{27.1}$$

where:

- status_i indicates whether endpoint i is still nonterminal, confirmed atomic, or confirmed composite;
- κ_i records the least-prime-factor responsibility of endpoint i;
- $\kappa(r)$ records the relation first killer;
- $\mathsf{Status}(r)$ records whether the relation is candidate, dead, or permanent.

This interface applies to

- fixed sums;
- fixed gaps;
- prime tuples;
- finite affine relations;
- finite polynomial bundles;
- arbitrary finite families of linear relations.

The core state of a relation sieve is not the survivor count, but the **complete relation instance and its terminal history**.

Chapter 28 — From Finite Terminality to Unified Prime Relations Theory

This part establishes four levels that cannot replace one another.

The first level is fact:

$$\boxed{\boldsymbol{I}(n)}$$

gives the fixed identity of an integer.

The second level is relation:

$$\boxed{r_{\Theta,x} = (f_1, \dots, f_m)}$$

specifies which positions form one object.

The third level is finite terminality:

$$\boxed{\tau(r) < \infty.}$$

Every finite relation can acquire a final local identity in a finite horizon.

The fourth level is quantification:

$$\boxed{\forall, \quad \exists, \quad \exists^{\infty}, \quad \forall k \exists \cdots}$$

determining how a local relation extends into a global proposition.

Unified Prime Relations Theory does not collapse these four layers into one.

Its strength lies precisely in strict stratification:

$$\boxed{\text{fixed identity} \Rightarrow/ \text{ relation automatically holds,}}$$

$$\boxed{\text{decidable relation} \Rightarrow/ \text{ universal statement automatically closes,}}$$

but at the same time

$$\boxed{\text{fixed identity} + \text{relation reconstruction} + \text{finite terminality} + \text{quantifier closure}}$$

provides a complete route for unified treatment of prime-relation problems.

Conclusion — The Mathematical Entry Point to an Infinite Relation Is Finite Terminality

Part I established

how atoms are generated, how relations are born, and how history accumulates.

This part further establishes

how relations are faithfully preserved and how they acquire final identity inside finite horizons.

Together, the matrix dynamics becomes

persistence of old relations + continuous birth of new relations + eventual freezing of every finite relation.

Infinity therefore need not be manipulated all at once as a completed object.

It is formed continuously by a sequence of finite, complete, verifiable, and mutually compatible relation layers:

$$F_1 \hookrightarrow F_2 \hookrightarrow F_3 \hookrightarrow \cdots, \qquad F_\infty = \varinjlim_t F_t.$$

Thus

A finite moving horizon is not a truncation of an infinite relation;
it is the entry point through which an infinite relation acquires rigorous mathematical identity.

But this statement must be retained together with another:

The fixed spectrum is only the factual foundation;
relation reconstruction is what turns facts into relational propositions.

The true power of Unified Prime Relations Theory therefore does not come from ignoring differences among relations, but from studying those differences clearly enough.

Once relation positions, endpoint correspondence, birth and death, terminal horizons, readout labels, and outer quantifiers have all been expressed precisely, fixed sums, fixed gaps, windows, finite patterns, affine relations, polynomial relations, consecutive primes, and complete factor identities can be organized in batch inside the same matrix without losing the structure that actually determines each proposition.

This yields the final global expression of this part:

fixed identity spectrum
⇓
relation instance
⇓
relation-faithful state
⇓
finite moving horizon
⇓
genuine death or terminal promotion
⇓
permanent relation skeleton
⇓
relation slice
⇓
quantifier decision.

This chain places “fact,” “relation,” “finite,” and “infinite” into one rigorous order.

It neither mystifies relations nor dissolves them into simple queries against the fixed spectrum.

It preserves every structure that prime relations genuinely need to preserve.

Part III Relation Slices, Derived Geometry, Quantifier Structures, and Batch Proof

— From the Sum–Difference–Product Relation Plane to a General Relational Language, Universal Decision, and Infinitude

Central theme of this part

The fixed spectrum allows different prime-number problems to share the same factual foundation; relation reconstruction ensures that this unification does not destroy the problems themselves.

Accordingly, the third layer of Unified Prime Relations Theory does not reduce different problems to the same number. It first preserves the genuine geometric relations among objects and then writes these relations in a common **relational language**:

relation geometry → relation slices → relational syntax → quantifiers.

Binary atomic relations first form a lossless sum–difference plane. Fixed sum, fixed difference, and fixed product form, respectively, linear leaves, linear leaves, and hyperbolic leaves in this same plane; endpoint identities can in turn be recovered from the unique intersections of oppositely oriented shells. Only after this layer of relation geometry has been preserved can a unified syntax avoid flattening distinct problems.

position maps + local relation predicates + outer quantifiers.

At this level, Goldbach, Twin, Polignac, Sophie Germain, prime tuples, Green–Tao, Legendre, Oppermann, Brocard, Andrica, Firoozbakht, Mersenne, Bunyakovsky, Schinzel H, Grimm, Erdős–Woods, and related problems no longer require separate underlying objects.

But unification never means that relations may be erased.

A higher-order expression must be recoverable to its lower-order witnesses;
any compression that cannot recover the target relation cannot replace that relation.

On this basis, proofs carried out one problem at a time can be elevated to **batch proofs**: a single uniform certificate can cover an entire parameter family. Universality and infinitude are no longer conflated; each receives its own exact decision form.

Abstract

The first two parts established two basic classes of structure.

The first is the fixed arithmetic identity spectrum

$$\boldsymbol{I}(n),$$

which assigns to each integer its fixed prime/composite status, least prime factor, multiplicative depth, complete prime-factor set, and related arithmetic identity data.

The second is relation reconstruction together with finite terminality. Every finite relation can retain its position maps, endpoints, identities, and history, and can receive terminal certification within a finite moving horizon; once confirmed, a relation is preserved permanently in all larger horizons.

This part establishes the third layer: **a unified relational language and quantifier structure**.

We begin with the binary atomic relation matrix. For atoms

$$a \leq b,$$

define

$$m = a + b + 1, \qquad h = b - a.$$

For the corresponding ordinary odd primes

$$p = 2a + 1, \qquad q = 2b + 1,$$

we have

$$p + q = 2m, \qquad q - p = 2h,$$

with inverse formulas

$$p = m - h, \qquad q = m + h.$$

Thus every unordered pair of odd primes corresponds uniquely to a point

$$(m, h)$$

in the relation plane.

Fixing m gives a fixed-sum slice; fixing h gives a fixed-difference slice; fixing the larger atom gives an oblique relation star. Goldbach and Twin are therefore readings in different directions of the same binary atomic relation plane.

The same relation plane also carries a natural product coordinate:

$$(m - h)(m + h) = m^2 - h^2.$$

Hence a fixed product M corresponds to the hyperbolic leaf

$$\boxed{m^2 - h^2 = M.}$$

If the total multiplicative depth of a relation is defined by

$$\mathcal{T}(m, h) = \Omega(m - h) + \Omega(m + h),$$

then, because $\Omega(xy) = \Omega(x) + \Omega(y)$,

$$\boxed{\mathcal{T}(m, h) = \Omega(m^2 - h^2).}$$

Consequently, on a fixed-product leaf,

$$\boxed{m^2 - h^2 = M \Rightarrow \mathcal{T}(m, h) = \Omega(M).}$$

Therefore

sum, difference, and product are not three separate relation spaces;
they are three foliations of the same binary atomic relation plane.

More generally, for arbitrary odd integer endpoints x, y, define two families of relation shells:

$$L_x^- = \{(m, h) : m - h = x\}, \qquad L_y^+ = \{(m, h) : m + h = y\}.$$

The two shells intersect uniquely at

$$\boxed{\left(\frac{x+y}{2}, \frac{y-x}{2}\right).}$$

Thus a binary relation instance is equivalent to the unique intersection of two oppositely oriented shells. Its multiplicative-depth vector is precisely

$$\boxed{(\Omega(x), \Omega(y)).}$$

The sum–difference plane also possesses two orthogonal generating directions:

$$\left(\frac{x+y}{2}, \frac{y-x}{2}\right) = \frac{x}{2}(1,-1) + \frac{y}{2}(1,1),$$

and

$$(1,-1) \cdot (1,1) = 0.$$

Hence a gap between neighboring atoms is not merely a distance in the one-dimensional atomic sequence; it also becomes a local step size of the relation grid along its two generating directions.

Next define the finite Goldbach witness field

$$W_K(m) = \{a \in A_K : m - 1 - a \in A_K\}.$$

If

$$T_{K,h} = \#\{b \in A_K : b + h \in A_K\},$$

then the exact identity

$$\boxed{\sum_m |W_K(m) \cap W_K(m+h)| = K\, T_{K,h}}$$

holds. For $h = 1$, the right-hand side is exactly K times the number of Twin pairs in the finite range. More generally, for a finite offset pattern

$$H = \{0, h_1, \ldots, h_r\},$$

if

$$T_K(H) = \#\{b : b + h \in A_K \text{ for every } h \in H\},$$

then

$$\boxed{\sum_m \left|\bigcap_{h \in H} W_K(m+h)\right| = K\, T_K(H).}$$

Thus finite prime constellations can be read exactly as persistent multi-column overlaps of the same fixed-sum witness field. No higher-order average replaces the lower-order relation here: both sides of the identity still correspond pointwise to the same lower-order atomic witnesses.

The relational language is then extended to an arbitrary finite family of position maps

$$f_1, \ldots, f_r.$$

A relation instance is written in the normal form

$$R_\Theta(x) = \mathcal{B}(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_r(x;\Theta))),$$

where Θ is a parameter, x is an anchor, and $\mathcal{B}$ is a finite relation predicate. The proposition then places its quantifier outside:

$$\boxed{\mathcal{Q}_{\Theta,x}[R_\Theta(x)].}$$

This normal form covers fixed sums, fixed differences, affine relations, finite patterns, arithmetic-progression chains, windows, polynomial maps, exponential maps, adjacent-prime indices, and complete prime-factor-set relations.

Green–Tao arithmetic progressions are developed as a complete high-order example. Define

$$C_k(a,h) = \prod_{j=0}^{k-1} \alpha\,(a+jh).$$

This quantity satisfies the concatenation into binary equal-gap links

$$C_k(a,h) = \prod_{j=0}^{k-2} E_h\,(a+jh), \qquad E_h(x) = \alpha(x)\alpha(x+h),$$

and the recurrence

$$C_{k+1}(a,h) = C_k(a,h)\alpha(a+kh).$$

Any k-term prime arithmetic progression can be folded by central reflection onto the same fixed sum:

$$p_j + p_{k-1-j} = p_0 + p_{k-1}.$$

Hence

$$G(N) \geq \left\lceil \frac{k}{2} \right\rceil.$$

Arithmetic progressions also have exact local forbidden classes. For an odd prime q, let $\rho_q(k,h)$ denote the number of starting residue classes forbidden at the q-layer. Then

$$\rho_q(k,h) = \begin{cases} 1, & q \mid h, \\ \min(k,q), & q \nmid h. \end{cases}$$

Therefore, for a finite set of primes S, the number of admissible starting classes is

$$\prod_{q\in S} (\,q - \rho_q(k,h)).$$

In particular, if the first term of a complete k-term equal-gap prime progression exceeds k, then every odd prime

$$q \leq k$$

must divide the step h. This explains why the step sizes of arbitrarily long prime arithmetic progressions must absorb progressively more low-order multiplicative structure.

Finally, this part distinguishes eight quantifier structures rigorously:

1. finite readout;
2. infinite recurrence;
3. parameterwise universal fibers;
4. window universality;
5. arbitrary length;
6. parameter infinitude;
7. indexwise universal predicates;
8. identity-block separation.

It also establishes the general form of batch proof. If a parameter family admits a uniform certificate map

$$\mathcal{C}: \Theta \to \mathcal{S},$$

satisfying coverage and correctness, the same proof covers the entire parameter family at once. The decisive issue in universality and infinitude is not merely to increase the amount of computation, but to establish a uniform closure law that can be repeated, propagated, or accumulated.

The final structure of this part is therefore

fixed spectrum → lower-order relation witnesses → unified relational syntax → quantifier classification → batch certificates

Keywords: relation slice; sum–difference–product relation plane; fixed-product hyperbolic leaf; relation shell; orthogonal generation; lower-order witness; fixed sum; fixed difference; finite atomic constellation; Green–Tao; relation normal form; quantifier classification; batch proof; universal decision; infinite recurrence

Part I The Binary Atomic Relation Plane and Derived Geometry

Chapter 1 The Binary Atomic Relation Plane: A Common Matrix for Sum and Difference

Part I proved that every unordered binary atomic relation

$$\{a, b\}, \qquad a \le b,$$

can be represented losslessly by sum–difference coordinates.

Define

$$\boxed{m = a + b + 1, \qquad h = b - a.} \tag{1.1}$$

For the corresponding ordinary odd primes

$$p = 2a + 1, \qquad q = 2b + 1,$$

we then have

$$\boxed{p + q = 2m,} \tag{1.2}$$

and

$$\boxed{q - p = 2h.} \tag{1.3}$$

Conversely,

$$\boxed{p = m - h, \qquad q = m + h.} \tag{1.4}$$

Hence, subject only to the appropriate lattice-integrality conditions, a relation point

$$(m, h)$$

uniquely recovers the original unordered prime pair.

Definition 1.1 Binary Atomic Relation Plane

Define

$$\boxed{\Gamma = \{(m, h): m - h,\ m + h \text{ are both odd primes}\}.} \tag{1.5}$$

Equivalently, in atomic coordinates,

$$\boxed{\Gamma = \left\{(m, h): \frac{m - h - 1}{2}, \frac{m + h - 1}{2} \in A\right\}.} \tag{1.6}$$

Here

$$m \ge 3, \qquad h \ge 0.$$

Every

$$(m, h) \in \Gamma$$

corresponds to the unique unordered prime pair

$$\boxed{\{m - h, m + h\}.} \tag{1.7}$$

Thus Γ is not a statistical diagram. It is a lossless coordinatization of the binary atomic relation itself.

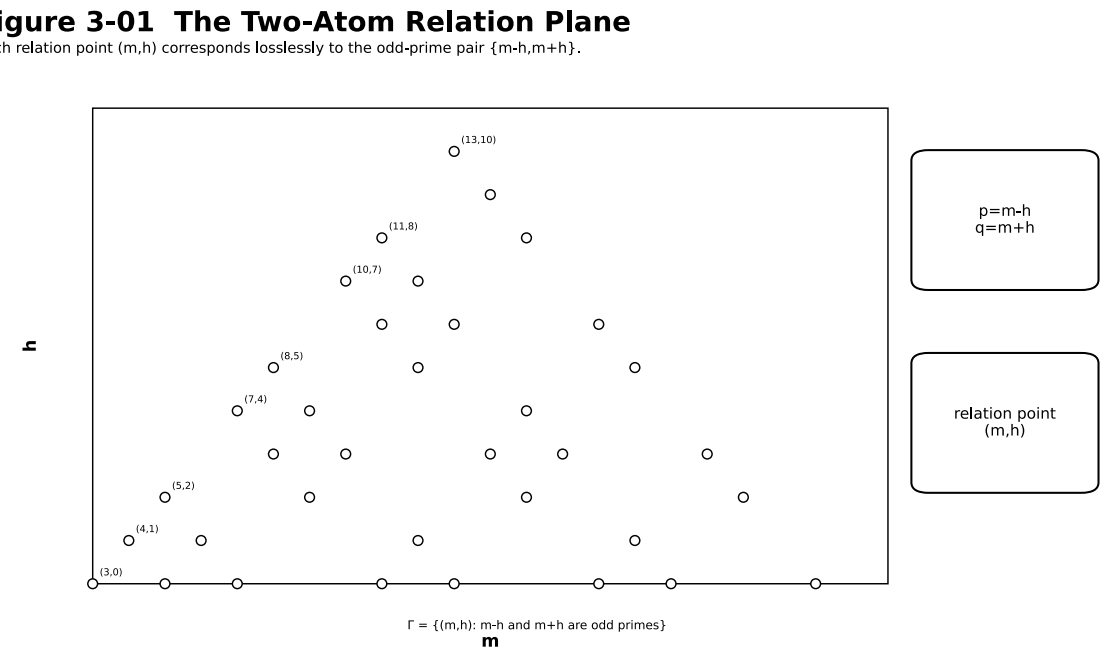


Figure 3-01. The Two-Atom Relation Plane. Each relation point $(m, h) \in \Gamma$ corresponds losslessly to the unique unordered odd-prime pair $\{m - h, m + h\}$. Thus (m, h) is not a statistical coordinate system but a two-dimensional coordinate representation of the binary atomic relation itself.

Chapter 2 Three Natural Directions in the Relation Plane

The same two-dimensional relation plane admits three different but equally natural directions of readout.

2.1 Fixed Sum: Vertical Slices

For fixed m, define

$$\boxed{\Gamma_m = \{h: (m,h) \in \Gamma\}.} \tag{2.1}$$

Every

$$h \in \Gamma_m$$

gives an unordered representation by odd primes,

$$2m = (m-h) + (m+h).$$

Define the fixed-sum multiplicity

$$\boxed{G(2m) = |\Gamma_m|.} \tag{2.2}$$

The local relational form of binary Goldbach is therefore

$$\boxed{\Gamma_m \neq \varnothing.} \tag{2.3}$$

Its universal form is

$$\boxed{\forall m \geq 3, \qquad \Gamma_m \neq \varnothing.} \tag{2.4}$$

The important point is that fixing the sum does not create a new class of relations. It simply reads one vertical fiber of the relation plane.

2.2 Fixed Difference: Horizontal Slices

For fixed h, define

$$\boxed{\Gamma^{[h]} = \{m: (m,h) \in \Gamma\}.} \tag{2.5}$$

Every

$$m \in \Gamma^{[h]}$$

corresponds to a pair of odd primes with difference

$$2h.$$

Define

$$\boxed{T(2h) = |\Gamma^{[h]}|.} \tag{2.6}$$

When

$$h = 1,$$

we have

$$q - p = 2,$$

so the Twin relation is precisely the relation points on the horizontal line

$$h = 1.$$

The infinite Twin proposition has the exact form

$$\boxed{|\Gamma^{[1]}| = \infty.} \tag{2.7}$$

Thus

Goldbach reads vertical fibers;

Twin reads the unit horizontal fiber.

(2.8)

2.3 Fixed Larger Atom: An Oblique Relation Star

Fix the larger prime

$$q = 2b + 1.$$

With each historical prime

$$p = 2a + 1, \qquad a \le b,$$

it forms the relation point

$$m = a + b + 1, \qquad h = b - a.$$

Hence

$$m + h = 2b + 1 = q.$$

Therefore all relation points formed by the same new prime with all historical primes lie on the oblique line

$$\boxed{m + h = q.} \tag{2.9}$$

If $q = p_k$ is the k-th odd prime, then this oblique relation star has exactly k points.

Therefore

A new atom generates an oblique relation star in the sum–difference plane.

(2.10)

These k points simultaneously carry

- k fixed-sum relations;
- $k - 1$ positive-difference relations;
- one self-pairing relation.

They are not three different events, but three readings of the same collection of atomic pairs.

Figure 3-02 Three Natural Directions: Sum, Difference, and Relation Star

In relation-plane coordinates (m, h), $p = m - h$, $q = m + h$. Fixed m, fixed h, and fixed larger prime $q = m + h$ are three readings of the same plane.

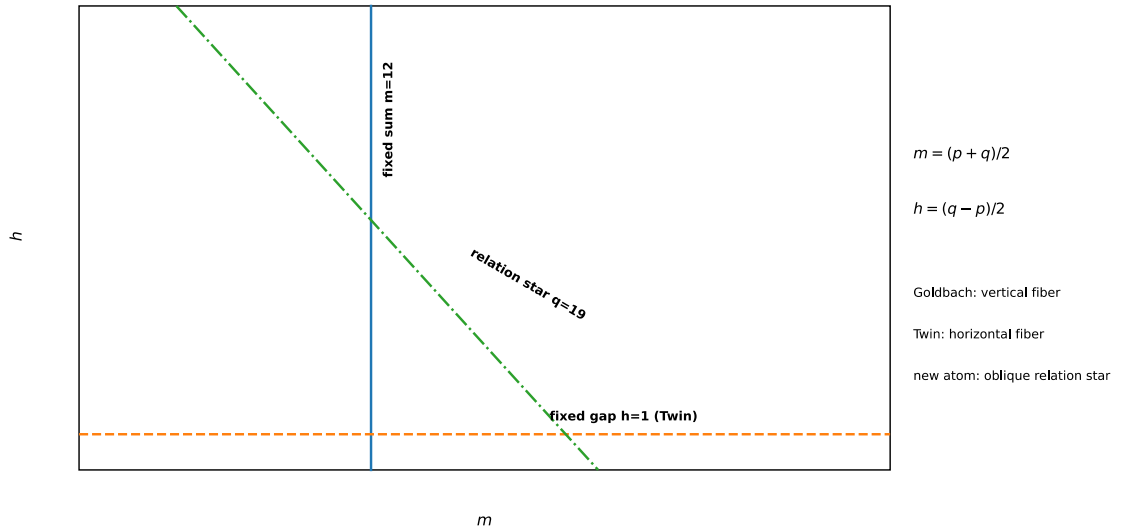


Figure 3-02. Three Natural Directions: Sum, Difference, and Relation Star. In the sum–difference relation plane (m, h), fixed m gives a Goldbach-type vertical fixed-sum fiber, fixed h gives a horizontal fixed-gap fiber, and fixing the larger prime $q = m + h$ gives an oblique relation star.

2.4 Fixed Product: Hyperbolic Leaves of the Same Plane

Sum and difference do not exhaust the information in the binary atomic relation plane.

From

$$p = m - h, \qquad q = m + h,$$

we immediately obtain

$$\boxed{pq = (m - h)(m + h) = m^2 - h^2.} \tag{2.11}$$

Thus the fixed-product condition

$$pq = M$$

corresponds in the (m, h)-plane to

$$\boxed{m^2 - h^2 = M.} \tag{2.12}$$

This is a family of hyperbolic leaves.

Fixed sum reads

$$m = \text{constant},$$

fixed difference reads

$$h = \text{constant},$$

and fixed product reads

$$m^2 - h^2 = \text{constant}.$$

Hence

sum, difference, and product are not three separate relation spaces;
they are three foliations of the same relation plane.

(2.13)

This point is fundamental.

The sum is one linear combination.

The difference is another linear combination.

The product recombines the two linear coordinates into a multiplicative relation.

None of these operations changes the endpoints or their identities. They only read different invariants from the same lossless relation point.

Figure 3-03 Hyperbolic Leaves for Fixed Products

Fixed sum, fixed gap, and fixed product are different leaves of the same (m,h) plane.

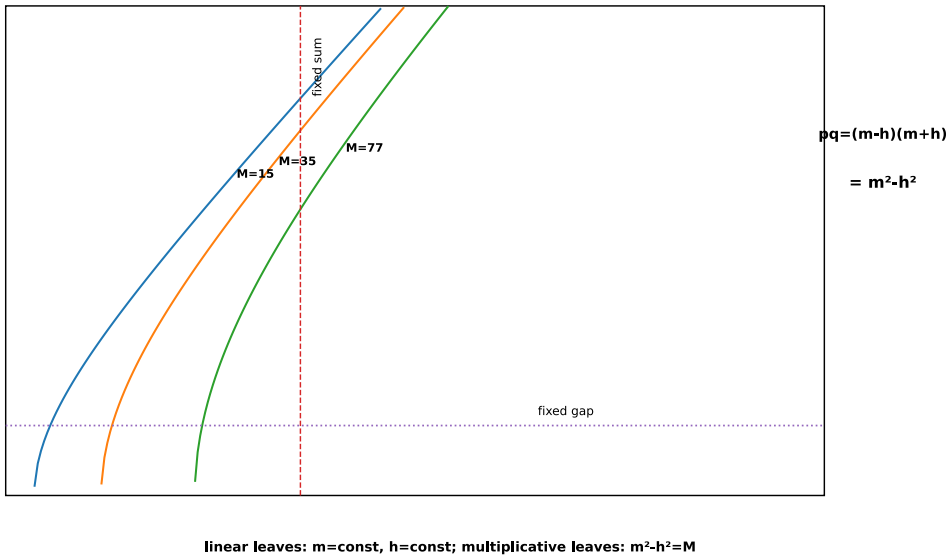


Figure 3-03. Hyperbolic Leaves for Fixed Products. From $p = m - h$ and $q = m + h$, one has $pq = m^2 - h^2$. Hence fixed sum $m = \text{const}$, fixed gap $h = \text{const}$, and fixed product $m^2 - h^2 = M$ are three families of leaves in the same relation plane, not three different object spaces.

2.5 Total Multiplicative-Depth Identity

For every positive integer n, let

$$\Omega(n)$$

denote the total number of prime factors counted with multiplicity.

Define the total multiplicative depth of the relation point (m, h) by

$$\boxed{\mathcal{T}(m,h) = \Omega(m-h) + \Omega(m+h).} \tag{2.14}$$

Because

$$\Omega(xy) = \Omega(x) + \Omega(y),$$

we obtain the following.

Theorem 2.1 Total Multiplicative-Depth Identity

$$\boxed{\mathcal{T}(m,h) = \Omega(m^2 - h^2).} \tag{2.15}$$

Proof

Since

$$m^2 - h^2 = (m-h)(m+h),$$

and Ω is additive under multiplication,

$$\begin{aligned}\Omega(m^2 - h^2) &= \Omega((m-h)(m+h)) \\ &= \Omega(m-h) + \Omega(m+h) \\ &= \mathcal{T}(m,h).\end{aligned}$$

This proves the theorem.

Corollary 2.2 Depth Conservation on a Fixed-Product Leaf

If

$$m^2 - h^2 = M,$$

then

$$\boxed{\mathcal{T}(m,h) = \Omega(M).} \tag{2.16}$$

Thus every relation point on the same fixed-product hyperbolic leaf has the same **total multiplicative depth**.

What is constant is the sum of the two endpoint depths; the individual endpoint depths need not be equal.

For example,

$$\Omega(m-h) = 1, \qquad \Omega(m+h) = 3,$$

and

$$\Omega(m-h) = 2, \qquad \Omega(m+h) = 2$$

can have the same total depth 4, although they plainly belong to different relation identities.

Total depth is therefore a genuine invariant, but it cannot replace endpoint-level identity.

This again illustrates a basic discipline of the book:

An invariant may compress a relation;
after compression, one must still know which information was preserved and which was not.

(2.17)

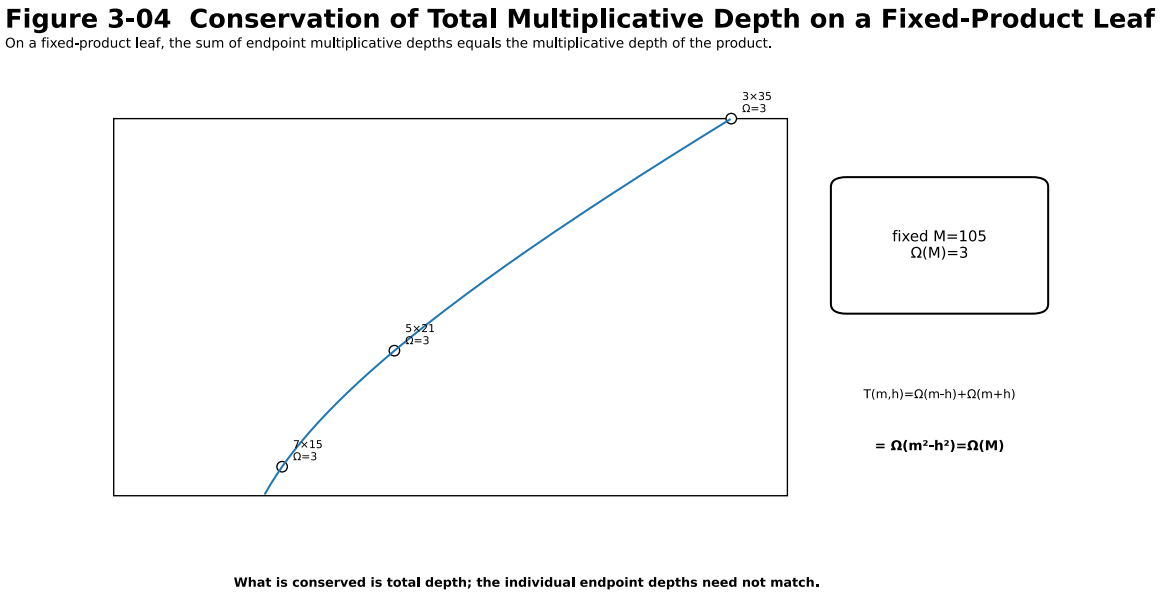


Figure 3-04. Conservation of Total Multiplicative Depth on a Fixed-Product Leaf. The total multiplicative depth $\mathcal{T}(m,h) = \Omega(m-h) + \Omega(m+h)$ satisfies $\mathcal{T}(m,h) = \Omega(m^2 - h^2)$. Hence on a fixed-product leaf $m^2 - h^2 = M$, the total depth is constantly $\Omega(M)$; what is conserved is the sum of endpoint depths, not each endpoint depth separately.

Chapter 3 Shell Intersections, Orthogonal Generation, and Reflection–Translation

Sum, difference, and product show that the same relation point can be read along different foliations.

Now reverse the question:

> *Given two endpoints, how do they generate a unique point in the relation plane?*

The answer is provided by two families of relation shells.

3.1 Two Families of Endpoint Shells

For an odd integer $x \geq 3$, define the left-endpoint shell

$$\boxed{L_x^- = \{(m,h): m - h = x\}.} \tag{3.1}$$

For an odd integer $y \geq 3$, define the right-endpoint shell

$$\boxed{L_y^+ = \{(m,h): m + h = y\}.} \tag{3.2}$$

The shell L_x^- keeps the left endpoint fixed;

L_y^+ keeps the right endpoint fixed.

Thus each relation point belongs simultaneously to one negative-oriented shell and one positive-oriented shell.

Theorem 3.1 Shell-Intersection Bijection

If x, y have the same parity, then

$$\boxed{L_x^- \cap L_y^+ = \left\{\left(\frac{x+y}{2}, \frac{y-x}{2}\right)\right\}.} \tag{3.3}$$

Proof

If

$$(m,h) \in L_x^- \cap L_y^+,$$

then

$$m - h = x, \qquad m + h = y.$$

Adding and subtracting gives

$$m = \frac{x+y}{2}, \qquad h = \frac{y-x}{2}.$$

Hence there is at most one intersection point.

Conversely, substituting this pair gives

$$m - h = x, \qquad m + h = y,$$

so the point lies on both shells.

This proves the theorem.

Therefore

$$\boxed{\text{binary relation instance} \leftrightarrow \text{unique intersection of oppositely oriented shells.}} \tag{3.4}$$

This gives the relation matrix a direct two-dimensional geometric realization.

Figure 3-05 Shell-Intersection Bijection

A fixed-left-endpoint shell and a fixed-right-endpoint shell meet at exactly one point, which is the relation instance.

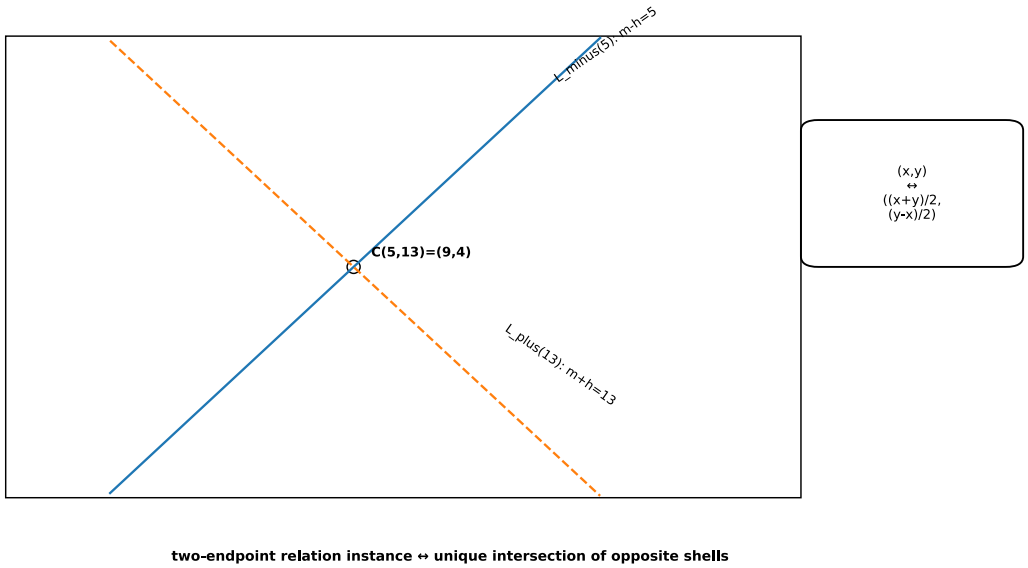


Figure 3-05. Shell-Intersection Bijection. The shell L_x^- of a fixed left endpoint x and the shell L_y^+ of a fixed right endpoint y meet at the unique point $C(x,y) = ((x+y)/2, (y-x)/2)$. Thus a two-endpoint relation instance is in bijection with the unique intersection of oppositely oriented shells.

3.2 Factorization of Shell Depth

Define the endpoint-depth vector

$$\boxed{\Delta(C(x,y)) = (\Omega(x), \Omega(y)),} \tag{3.5}$$

where

$$C(x,y) = \left(\frac{x+y}{2}, \frac{y-x}{2}\right).$$

Then

- the first depth component is fixed at $\Omega(x)$ on L_x^-;
- the second depth component is fixed at $\Omega(y)$ on L_y^+.

In particular, if

$$x, y \in \mathbb{P}_{\text{odd}},$$

then

$$\boxed{\Delta(C(x,y)) = (1,1).} \tag{3.6}$$

Thus the binary atomic relation plane is exactly the set of all shell intersections of depth $(1,1)$.

3.3 Shell Geometry of the Birth of New Atoms

Suppose there are already K atoms and that M new atoms are then incorporated at once.

The new unordered relations divide into two classes.

The first consists of old–new relations:

$$K \times M.$$

They form a rectangular region in the relation matrix.

The second consists of new–new relations, including self-pairings:

$$\frac{M(M+1)}{2}.$$

They form an upper-triangular region.

Hence the total number of new relations is

$$\boxed{KM + \frac{M(M+1)}{2}.} \tag{3.7}$$

Geometrically,

$$\boxed{\text{new relations} = \text{old–new rectangle} + \text{new–new triangular region.}} \tag{3.8}$$

Each new relation is also exactly a newly appearing intersection of a pair of oppositely oriented shells.

Therefore

$$\boxed{\text{growth of the relation matrix} \Leftrightarrow \text{growth of shell intersections.}} \tag{3.9}$$

“A new atom gives birth to a full row of relations” and “a new shell enters and intersects the historical shells of opposite orientation one by one” are the matrix language and geometric language for the same event.

3.4 Orthogonal Generating Directions

The shell-intersection coordinates can be written as

$$\boxed{C(x,y) = \left(\frac{x+y}{2}, \frac{y-x}{2}\right) = \frac{x}{2}(1,-1) + \frac{y}{2}(1,1).} \tag{3.10}$$

The two generating directions satisfy

$$\boxed{(1,-1) \cdot (1,1) = 0.} \tag{3.11}$$

Hence

the left and right endpoints generate the relation plane along two orthogonal directions.

(3.12)

If the left endpoint changes by δx, the relation point moves by

$$\frac{\delta x}{2}$$

along the direction

$$(1,-1).$$

If the right endpoint changes by δy, the relation point moves by

$$\frac{\delta y}{2}$$

along the direction

$$(1,1).$$

This makes local endpoint gaps directly into local geometric step sizes in the relation plane.

3.5 Prime Gaps as a Local Metric of the Relation Grid

Let consecutive odd primes be

$$p_n < p_{n+1},$$

with prime gap

$$g_n = p_{n+1} - p_n.$$

If the other endpoint y is fixed and the left endpoint changes from p_n to p_{n+1}, then the displacement of the relation point is

$$\boxed{\frac{g_n}{2}(1,-1).} \tag{3.13}$$

If the right endpoint changes instead, the displacement is

$$\boxed{\frac{g_n}{2}(1,1).} \tag{3.14}$$

Therefore

a prime gap is not merely a one-dimensional difference in the atomic sequence;
it is also a local metric step along the two orthogonal generating directions of the relation grid.

(3.15)

This also explains why prime gaps enter simultaneously into

- fixed-difference relations;
- translations of relation rows;
- widths of square horizons;
- adjacent-prime windows;
- local scales of the relation grid.

A prime gap is not an externally imposed coordinate label. It is a step size induced by the ordering of the atoms themselves in relation geometry.

3.6 Reflection and Translation: Two Self-Intersections of the Same Atomic Set

Let

$$A$$

be the set of all odd-prime atomic coordinates.

A fixed-sum relation can be written as a reflection self-intersection.

For $m \geq 3$, define the reflection

$$\boxed{\rho_m(a) = m - 1 - a.} \tag{3.16}$$

Then the fixed-sum witness set is

$$\boxed{A \cap \rho_m(A).} \tag{3.17}$$

The universal form of Goldbach is precisely

$$\boxed{\forall m \geq 3, \qquad A \cap \rho_m(A) \neq \varnothing.} \tag{3.18}$$

A fixed-difference relation can be written as a translation self-intersection.

Define

$$\boxed{\tau_h(a) = a + h.} \tag{3.19}$$

Then the witness set for fixed difference $2h$ is

$$\boxed{A \cap (A - h).} \tag{3.20}$$

Twin corresponds to

$$h = 1,$$

namely

$$\boxed{\#(A \cap (A - 1)) = \infty.} \tag{3.21}$$

Thus

$$\boxed{\begin{array}{r}\text{multiplication generates a rigid atomic set } A;\\ \text{sum and difference are self-intersections of } A \text{ under different transformations.}\end{array}} \tag{3.22}$$

This statement simultaneously preserves two facts:

- atomic identity comes from the fixed multiplicative structure;
- the truth of a relation comes from the concrete intersection between the atomic set and its transformed copy.

The fixed spectrum therefore does not dissolve relations.

It makes them expressible as exact self-intersections.

Figure 3-06 Orthogonal Generation and Reflection–Translation Self-Intersections

Endpoints generate the relation plane along two orthogonal directions; fixed sums and fixed gaps are reflection and translation self-intersections of the atomic set.

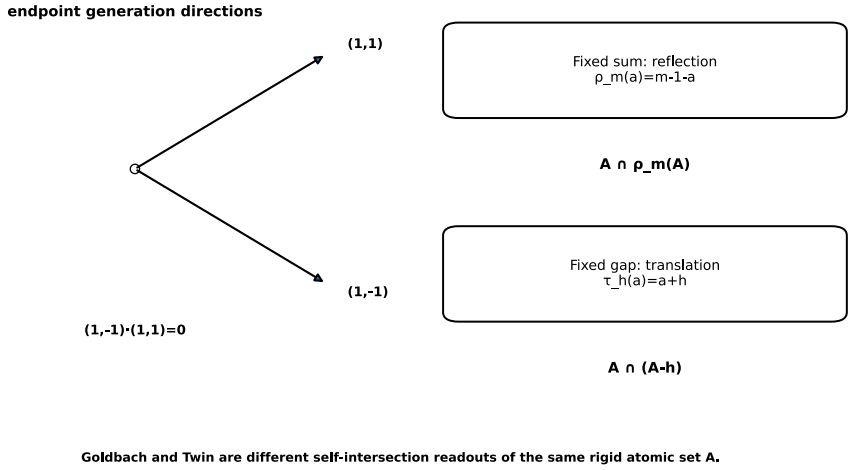


Figure 3-06. Orthogonal Generation and Reflection–Translation Self-Intersections. Left and right endpoints generate the relation plane along the orthogonal directions $(1,-1)$ and $(1,1)$. A fixed sum is the self-intersection of the atomic set A with its reflection $\rho_m(A)$, while a fixed gap is the self-intersection of A with a translated copy. Goldbach and Twin are therefore different relational readouts of the same rigid atomic set.

Chapter 4 Lower-Order Witness Discipline: Unification Cannot Be Purchased by Losing the Relation

If a higher-order statistic cannot recover the target lower-order witnesses, it cannot replace the target relation.

This discipline is especially clear for fixed-sum problems.

The lowest-order witness for Goldbach is a specific atom

$$a \in A$$

such that simultaneously

$$m - 1 - a \in A.$$

Thus

$$\boxed{a \in A \quad \text{and} \quad m - 1 - a \in A} \tag{4.1}$$

is a complete relation witness.

No matter whether one introduces

- total relation counts;
- average density;
- modular residue-class distributions;
- convolution averages;
- spectra;
- higher-order overlaps;

if the goal is to deduce a fixed-sum relation, the argument must ultimately recover such a concrete witness.

Therefore

no higher-order quantity may detach from the lower-order witness.

(4.2)

Here “higher order” does not prohibit the use of high-dimensional structure.

It imposes only the following requirement:

A high-dimensional structure must retain an explicit, provable, information-preserving interface to the lowest-order relation demanded by the target statement.

Part II Lower-Order Witness Fields and Finite Atomic Constellations

Chapter 5 The Finite Goldbach Witness Field

Take the first K atoms

$$A_K = \{a_1, \dots, a_K\}.$$

For an integer m, define the finite fixed-sum witness set

$$\boxed{W_K(m) = \{a \in A_K : m - 1 - a \in A_K\}.} \tag{5.1}$$

Then

$$|W_K(m)|$$

is the number of ordered left-end witnesses among the first K atoms for the target m.

If unordered relations are desired, one simply folds by the symmetry exchanging the two endpoints.

The essential point is not the count itself, but that

$$W_K(m)$$

retains every individual left-end atom.

Consequently, different target columns can be compared directly to determine whether the same left-end witness persists.

Figure 3-07 Finite Goldbach Witness Field

For a finite atomic set A_K, each fixed sum m has a witness column W_K(m).

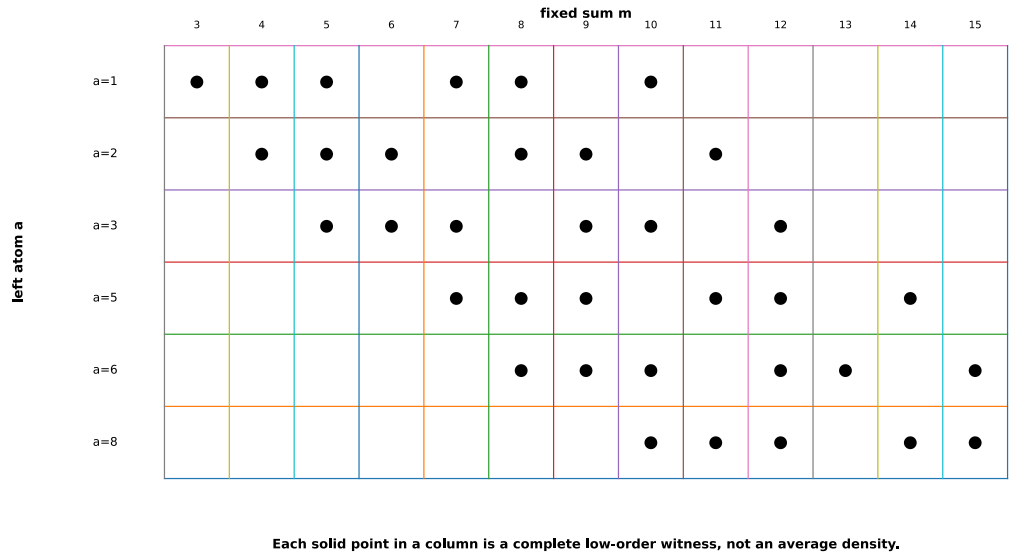


Figure 3-07. Finite Goldbach Witness Field. For a finite atomic set A_K, every fixed sum m has a witness column $W_K(m) = \{a \in A_K : m - 1 - a \in A_K\}$. Every solid point in the diagram is a concrete low-order atomic witness, not an average density or scalar approximation.

Chapter 6 Twin as Persistence Across Adjacent Fixed-Sum Witness Columns

Compare the adjacent columns

$$W_K(m)$$

and

$$W_K(m+1).$$

If

$$a \in W_K(m) \cap W_K(m+1),$$

then

$$b = m - 1 - a \in A_K,$$

and simultaneously

$$b + 1 = m - a \in A_K.$$

The corresponding ordinary odd primes are

$$2b + 1, \qquad 2b + 3,$$

which differ by 2.

Conversely, suppose

$$b, b + 1 \in A_K$$

form a Twin atomic pair. Choose any

$$a \in A_K$$

and set

$$m = a + b + 1.$$

Then

$$a \in W_K(m) \cap W_K(m+1).$$

Therefore

the same left-end fixed-sum witness persists from $2m$ to $2m + 2 \Leftrightarrow$ its two right-end partners a (6.1)

Equivalently,

Twin = persistence of a fixed-sum witness across adjacent even targets.

(6.2)

Figure 3-08 Twin as the adjacent persistence of a fixed-sum witness

The same left-end witness appears in both W_k(m) and W_k(m+1); exactly then the two columns form a Twin atomic pair under the curve condition.

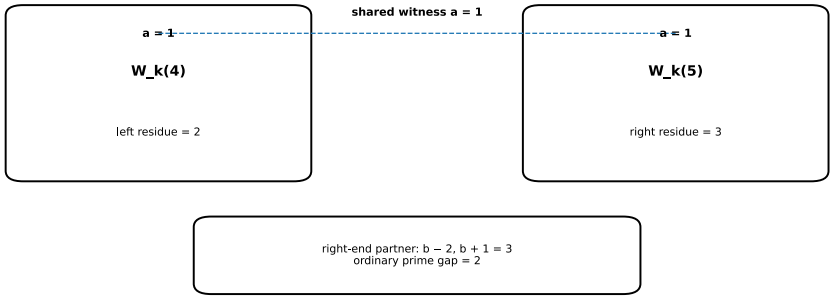


Figure 3-08. Twin as Persistence Across Adjacent Fixed-Sum Columns. If the same left witness a lies in both $W_K(m)$ and $W_K(m+1)$, its right partners are b and $b+1$, corresponding to ordinary primes differing by 2. The converse construction also holds, so Twin is read exactly as persistence of fixed-sum witnesses across adjacent even-sum columns.

Chapter 7 Exact Identity Between Fixed-Difference Counts and Witness Persistence

Define the finite atomic difference count

$$\boxed{T_{K,h} = \#\{b \in A_K : b + h \in A_K\}.} \tag{7.1}$$

It counts pairs of atomic coordinates separated by h; the corresponding ordinary prime difference is

$$2h.$$

Define also the total witness-overlap quantity

$$\boxed{\Omega_{K,h} = \sum_{m} |\, W_K(m) \cap W_K(m+h)|.} \tag{7.2}$$

Theorem 7.1 Witness-Persistence Identity

For every

$$h \geq 0,$$

we have

$$\boxed{\Omega_{K,h} = K\, T_{K,h}.} \tag{7.3}$$

Proof

If

$$a \in W_K(m) \cap W_K(m+h),$$

set

$$b = m - 1 - a.$$

Then

$$a \in A_K, \qquad b \in A_K, \qquad b + h \in A_K.$$

Thus every object counted on the left corresponds uniquely to a pair

$$(a, b),$$

with

$$a \in A_K, \qquad b, b + h \in A_K.$$

Conversely, choose any

$$a \in A_K$$

and any atomic pair of difference h,

$$b, b + h \in A_K.$$

Setting uniquely

$$m = a + b + 1$$

gives

$$a \in W_K(m) \cap W_K(m + h).$$

Hence the objects on the left are in bijection with

$$A_K \times \{b \in A_K : b + h \in A_K\}.$$

The first factor contains K elements, while the second contains $T_{K,h}$ elements.

Therefore

$$\Omega_{K,h} = K\, T_{K,h}.$$

This proves the theorem.

Corollary 7.2 Finite Twin Counting Identity

Set

$$h = 1.$$

Then

$$\boxed{\sum_m |\, W_K(m) \cap W_K(m+1)| = K\, T_{K,1}.} \tag{7.4}$$

Thus

$$\boxed{T_{K,1} = \frac{1}{K} \sum_m |\, W_K(m) \cap W_K(m+1)|.} \tag{7.5}$$

Therefore

within a finite range, the number of Twin prime pairs is exactly
the total overlap of adjacent columns of the fixed-sum witness field divided by the number of atoms.

(7.6)

This is an exact finite identity; it depends on neither probability nor an average model.

Chapter 8 Multi-Column Persistence and General Prime Constellations

Let

$$\boxed{H = \{0, h_1, \dots, h_r\}} \tag{8.1}$$

be a finite offset set.

Define the finite constellation count

$$\boxed{T_K(H) = \#\{b: b + h \in A_K \text{ for every } h \in H\}.} \tag{8.2}$$

We can then consider the multi-column overlap of the fixed-sum witness field:

$$\bigcap_{h \in H} W_K(m + h).$$

Theorem 8.1 Multi-Column Persistence Identity

$$\boxed{\sum_m \left| \bigcap_{h \in H} W_K(m + h) \right| = K\, T_K(H).} \tag{8.3}$$

Proof

If

$$a \in \bigcap_{h \in H} W_K(m + h),$$

set

$$b = m - 1 - a.$$

Because $0 \in H$,

$$b \in A_K.$$

For every

$$h \in H,$$

from

$$a \in W_K(m + h)$$

we obtain

$$m + h - 1 - a = b + h \in A_K.$$

Thus every object on the left corresponds uniquely to an element

$$a \in A_K$$

and to a constellation starting point b satisfying

$$b + h \in A_K \quad (h \in H).$$

The reverse construction is identical to that in Theorem 7.1.

Hence the number of objects is

$$K\,T_K(H).$$

This proves the theorem.

Therefore

finite prime constellations are precisely persistent multi-column overlaps of the fixed-sum witness field.

(8.4)

The special case

$$H = \{0\}$$

gives a single-column witness;

$$H = \{0,1\}$$

gives Twin;

and more general H gives the corresponding finite prime pattern.

Thus the relation matrix does more than place Goldbach and Twin in the same coordinate system: it gives an exact quantitative interface between them.

Figure 3-09 Multi-Column Persistence and General Prime Constellations

A finite offset template H={0,h1,...,hr} corresponds to simultaneous overlap among several fixed-sum witness columns.

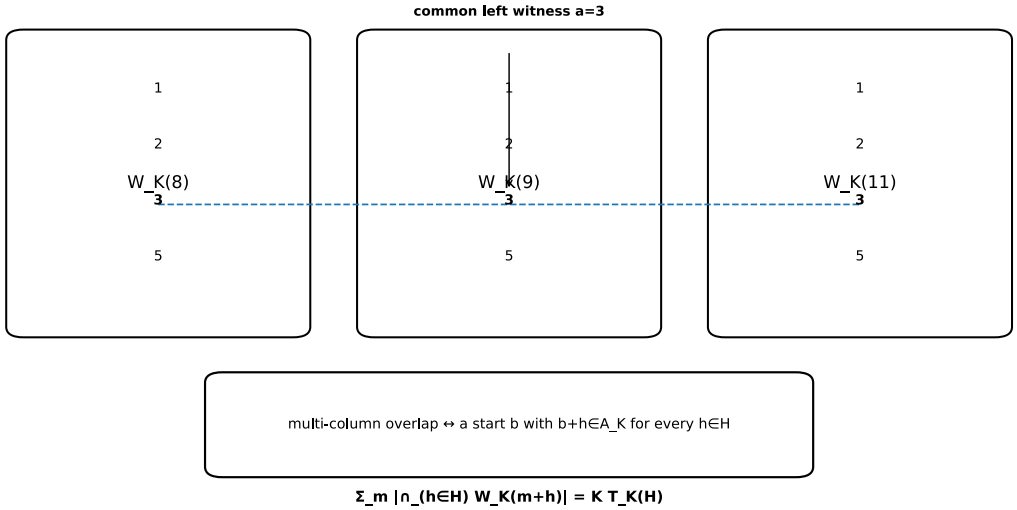


Figure 3-09. Multi-Column Persistence and General Prime Constellations. For a finite offset template $H = \{0, h_1, \dots, h_r\}$, a general prime constellation corresponds to simultaneous overlap among the fixed-sum witness columns $W_K(m+h)$, with the exact identity $\sum_m |\bigcap_{h\in H} W_K(m+h)| = K\,T_K(H)$.

Chapter 9 Temporal and Spatial Directions

The relation field possesses two independent natural directions.

Part I already gave the row update produced by atomic birth.

Suppose

$$A_K \to A_{K+1}$$

adds one new atom.

The fixed-sum multiplicity field changes with K; this is the **generative direction** of the relation field.

By contrast, Theorems 7.1 and 8.1 compare the overlap of different target columns; this is the **slice direction** of the relation field.

The relation field may therefore be regarded as having three coordinates:

$$\boxed{\begin{array}{lcl} K & : & \text{atomic generation and historical growth,} \\ m & : & \text{fixed-sum target,} \\ h & : & \text{column spacing and fixed difference.} \end{array}} \tag{9.1}$$

A new atom enters along the K-direction;

Goldbach is read along the m-direction;

prime gaps are read along the h-direction.

These three kinds of information belong to the same atomic relation field, not to three different underlying objects.

Part III General Relational Language and Relation Normal Form

Chapter 10 Definition of a General Relation Slice

The binary atomic relation plane gives the most transparent model of a slice.

To cover more general problems, “fixing a coordinate line” must be generalized to arbitrary finite position maps.

Definition 10.1 Relation Parameter

Let

$$\Theta$$

denote a collection of fixed relation parameters.

It may be

- a fixed sum N;
- a fixed difference d;
- affine coefficients (A, B);
- a finite offset pattern H;
- an arithmetic-progression length and step (k, d);
- a polynomial family $(f_1, \dots, f_r)$;
- a window index n;
- starting points (x, y) of two integer blocks.

Definition 10.2 Anchor

For fixed parameters Θ, let

$$x \in X_\Theta$$

be a relation anchor.

The anchor is the free position needed to generate a specific relation instance.

For example:

- in Goldbach, one may take $x = p$;
- in a fixed-difference relation, one may take $x = p$;
- in a prime tuple, one may take $x = n$;
- in Green–Tao, one may take $x = a$;
- in a polynomial relation, one may take $x = n$.

Definition 10.3 Position Maps

Given finitely many maps

$$\boxed{f_i : X_\Theta \times \Theta \to \mathbb{Z}, \qquad 1 \le i \le r,} \tag{10.1}$$

call

$$\boldsymbol{f}_\Theta = (f_1, \dots, f_r)$$

the **family of position maps** of the relation.

It answers the question:

> *Which integer positions in the fixed spectrum does a relation instance actually read?*

Definition 10.4 Local Relation Predicate

Given the position identities

$$\boldsymbol{I}(f_i(x;\Theta)),$$

define a finite predicate

$$\boxed{\mathcal{B}_\Theta}$$

acting on these identities together with any required numerical relations.

The local relation is written

$$\boxed{R_\Theta(x) = \mathcal{B}_\Theta(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_r(x;\Theta)))}. \tag{10.2}$$

If the only requirement is that all endpoints be prime, then

$$\boxed{R_\Theta(x) = \prod_{i=1}^{r} \alpha\,(f_i(x;\Theta))}. \tag{10.3}$$

Chapter 11 Unified Relation Normal Form

A complete proposition must also specify its outer quantifier.

We therefore define

$$\boxed{\mathcal{Q}_{\Theta,x}[R_\Theta(x)].} \tag{11.1}$$

Here

- $\boldsymbol{I}$: fixed arithmetic identity;
- f_i: relation positions;
- $\mathcal{B}$: local predicate;
- $\mathcal{Q}$: outer quantifier.

This gives the relation normal form used throughout the book:

$$\boxed{\mathcal{Q}_{\Theta,x}[\mathcal{B}_\Theta(\boldsymbol{I}(f_1(x;\Theta)),\dots,\boldsymbol{I}(f_r(x;\Theta)))].} \tag{11.2}$$

This formula separates a relation problem into three strict layers:

$$\boxed{\text{positions} \quad \rightarrow \quad \text{local relation} \quad \rightarrow \quad \text{global quantifier.}} \tag{11.3}$$

Figure 3-10 Unified Relation Normal Form: Positions → Local Relation → Global Quantifier
Part III compiles classical relation problems into one common relational syntax.

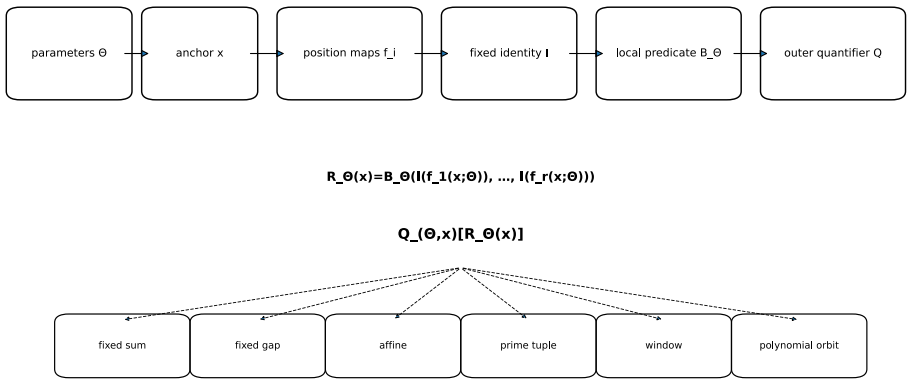


Figure 3-10. Unified Relation Normal Form: Positions → Local Relation → Global Quantifier. A complete proposition is decomposed into parameters Θ, anchor x, position maps f_i, fixed identity readout $\boldsymbol{I}$, a local predicate $\mathcal{B}$, and an outer quantifier $\mathcal{Q}$. Fixed sums, fixed gaps, affine relations, prime tuples, windows, and polynomial orbits are different parameterizations of this one normal form.

Chapter 12 Fixed Sum, Fixed Difference, and Consecutive Fixed Difference

12.1 Fixed Sum

For a given even integer N, set

$$f_1(p;N) = p, \qquad f_2(p;N) = N - p.$$

Define

$$\boxed{R_N(p) = \alpha(p)\alpha(N-p).} \tag{12.1}$$

The local relation asks only:

> *Does this value of p form a fixed-sum prime pair?*

The universal proposition is

$$\boxed{\forall N \in 2\mathbb{Z}_{\geq 4}, \quad \exists p: R_N(p) = 1.} \tag{12.2}$$

12.2 Fixed Difference

For a given even difference d, set

$$f_1(p) = p, \qquad f_2(p) = p + d.$$

Define

$$\boxed{W_d(p) = \alpha(p)\alpha(p+d).} \tag{12.3}$$

Twin corresponds to

$$d = 2.$$

12.3 Consecutive Fixed Difference

If one additionally requires p and $p + d$ to be consecutive primes, then the prime-free reading of the open interval between them must be preserved:

$$\boxed{E_d(p) = \prod_{p<n<p+d} (1 - \alpha(n)).} \tag{12.4}$$

The consecutive fixed-difference predicate is therefore

$$\boxed{P_d(p) = \alpha(p)\alpha(p+d)E_d(p).} \tag{12.5}$$

This shows that

$$\boxed{\text{fixed difference} \neq \text{consecutive fixed difference.}} \tag{12.6}$$

The two relations read the same fixed spectrum, but they have different local relation predicates.

Chapter 13 Weighted Reflections and Affine Relations

13.1 Lemoine / Lévy-Type Weighted Reflection

Given an odd integer N, require

$$N = p + 2q.$$

The position maps can be written

$$f_1(p; N) = p,$$

$$f_2(p; N) = \frac{N - p}{2}.$$

The local predicate is

$$\boxed{L_N(p) = \alpha(p)\alpha\left(\frac{N - p}{2}\right)} \tag{13.1}$$

with the additional requirement that

$$N - p$$

be a positive even integer.

Here the ordinary Goldbach reflection

$$p \mapsto N - p$$

is replaced by the weighted reflection

$$p \mapsto \frac{N - p}{2}.$$

13.2 General Affine Relations

For fixed integers A, B, define

$$\boxed{f_{A,B}(p) = Ap + B.} \tag{13.2}$$

Then

$$\boxed{R_{A,B}(p) = \alpha(p)\alpha(Ap + B).} \tag{13.3}$$

The Sophie Germain relation corresponds to

$$A = 2, \qquad B = 1.$$

Hence

$$\boxed{SG(p) = \alpha(p)\alpha(2p + 1).} \tag{13.4}$$

Affine relations show that “fixed sum” and “fixed difference” are not the boundary of the relational language.

Chapter 14 Finite Patterns and Prime Tuples

Let

$$\boxed{H = \{h_1, \dots, h_k\}} \tag{14.1}$$

be a finite offset set.

Define the position maps

$$f_j(n) = n + h_j.$$

The corresponding all-prime predicate is

$$\boxed{R_H(n) = \prod_{j=1}^{k} \alpha\,(n + h_j).} \tag{14.2}$$

Whenever

$$R_H(n) = 1,$$

the entire pattern

$$n + H$$

lies in the prime spectrum.

Twin corresponds to

$$H = \{0,2\}.$$

General prime constellations all have this form.

Therefore

preassigned conjecture names are not the boundary of the relational language;
the finite pattern itself is the mathematical object.

(14.3)

Part IV Higher-Order Affine Organization and Arithmetic Atomic Chains

Chapter 15 Green–Tao Arithmetic Atomic Chains

An especially important subclass of finite patterns is the arithmetic pattern.

In atomic coordinates, for

$$k \geq 2, \qquad a \geq 1, \qquad h \geq 1,$$

define the ordered equal-gap relation object

$$\boxed{\mathfrak{A}_k(a,h) = (a, a+h, \ldots, a+(k-1)h).} \tag{15.1}$$

Define its complete identity readout by

$$\boxed{C_k(a,h) = \prod_{j=0}^{k-1} \alpha\,(a+jh).} \tag{15.2}$$

We have

$$C_k(a,h) = 1$$

if and only if all k endpoints are atoms.

In ordinary prime coordinates,

$$\Phi(a+jh) = 2a + 1 + 2jh.$$

Thus the corresponding common difference between primes is

$$2h.$$

The entire chain is regarded here as a **single organized relation**, rather than as k unrelated prime events.

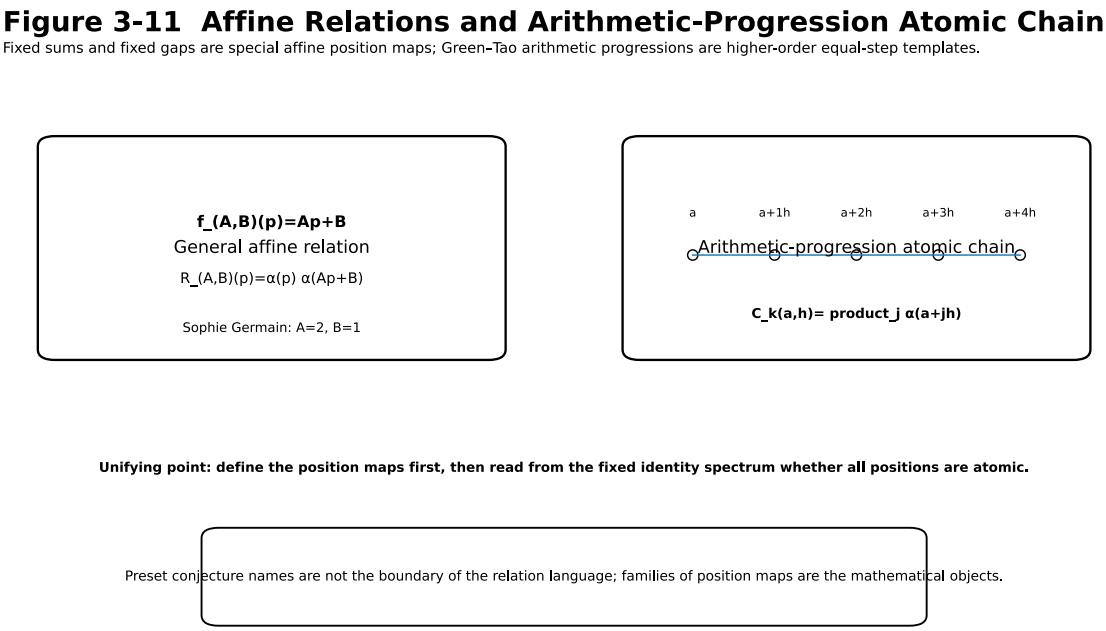


Figure 3-11. Affine Relations and Arithmetic-Progression Atomic Chains. A general affine relation is read through the position map $f_{A,B}(p) = Ap + B$, with Sophie Germain corresponding to $A = 2, B = 1$. A Green–Tao-type arithmetic atomic chain consists of $a, a + h, \ldots, a + (k - 1)h$. Together they show that preset conjecture names are not the boundary of the relation language; families of position maps are the mathematical objects.

Chapter 16 An Arithmetic Chain as a Concatenation of Equal-Gap Binary Atomic Relations

Define the equal-gap binary atomic relation

$$\boxed{E_h(x) = \alpha(x)\alpha(x+h).} \tag{16.1}$$

Theorem 16.1 Equal-Gap Chain Concatenation Theorem

For every

$$k \geq 2,$$

we have

$$\boxed{C_k(a,h) = \prod_{j=0}^{k-2} E_h\,(a+jh).} \tag{16.2}$$

Proof

Expanding the right-hand side gives

$$\alpha(a)\alpha(a+h)\cdot\alpha(a+h)\alpha(a+2h)\cdots\alpha(a+(k-2)h)\alpha(a+(k-1)h).$$

Every intermediate identity occurs twice.

Because

$$\alpha(n)^2 = \alpha(n),$$

the product reduces exactly to

$$\prod_{j=0}^{k-1} \alpha\,(a+jh) = C_k(a,h).$$

This proves the theorem.

Corollary 16.2 Chain-Length Recurrence

$$\boxed{C_{k+1}(a,h) = C_k(a,h)\,\alpha(a+kh).} \tag{16.3}$$

Thus extending an arithmetic chain by one term introduces only one new terminal atomic condition.

Chapter 17 Three-Term Chains and Fixed-Sum Reflection

When

$$k = 3,$$

$$C_3(a,h) = 1$$

gives three primes

$$p_0, \quad p_1, \quad p_2$$

satisfying

$$p_1 - p_0 = p_2 - p_1.$$

Therefore

$$\boxed{p_0 + p_2 = 2p_1.} \tag{17.1}$$

Theorem 17.1 Fixed-Sum Folding of a Three-Term Equal-Gap Chain

If

$$C_3(a,h) = 1,$$

then the same even integer

$$2p_1$$

has two unordered prime representations supplied by the chain:

$$\boxed{p_0 + p_2 = p_1 + p_1 = 2p_1.} \tag{17.2}$$

Conversely, if three odd primes

$$p_0 < p_1 < p_2$$

satisfy

$$p_0 + p_2 = 2p_1,$$

then they form a three-term prime arithmetic progression.

Chapter 18 Fixed-Sum Mirror Folding of an Arithmetic Chain of Arbitrary Length

Suppose

$$C_k(a,h) = 1,$$

and write

$$p_j = 2(a + jh) + 1.$$

Set

$$\boxed{N = p_0 + p_{k-1}.} \tag{18.1}$$

Then for every

$$0 \le j \le \left\lfloor \frac{k-1}{2} \right\rfloor,$$

we have

$$\boxed{p_j + p_{k-1-j} = N.} \tag{18.2}$$

Theorem 18.1 Arithmetic-Chain Mirror-Folding Theorem

A prime arithmetic progression of length k contributes at least

$$\boxed{\left\lceil \frac{k}{2} \right\rceil} \tag{18.3}$$

distinct unordered prime representations to the same fixed sum N.

Therefore

$$\boxed{G(N) \ge \left\lceil \frac{k}{2} \right\rceil.} \tag{18.4}$$

If k is odd, then the central term

$$p_{(k-1)/2}$$

is itself prime and

$$\boxed{N = 2p_{(k-1)/2}.} \tag{18.5}$$

Thus a higher-order relation can project multiple genuine witnesses onto a lower-order fixed-sum slice.

But the higher-order relation does not replace the lower-order relation:

each Goldbach witness remains recoverable as a concrete pair of prime endpoints.

Chapter 19 Square Horizons for Arithmetic Chains

Let

$$r < s$$

be consecutive odd primes.

Let the ordinary odd endpoints be

$$n_j = 2(a + jh) + 1.$$

Assume

$$n_0 > r,$$

and that the largest endpoint satisfies

$$r^2 \le n_{k-1} < s^2.$$

Define

$$\boxed{P_r = \prod_{\substack{q \le r \\ q \text{ prime}}} q\,.} \tag{19.1}$$

Theorem 19.1 Endpoint-Bounded Square Horizon for an Organized Chain

Under the above assumptions,

$$\boxed{C_k(a,h) = 1 \Leftrightarrow \gcd\left(\prod_{j=0}^{k-1} n_j\,, P_r\right) = 1.} \tag{19.2}$$

Proof

If all n_j are prime and

$$n_0 > r,$$

then none of the n_j is divisible by a prime

$$q \le r,$$

so the greatest common divisor is 1.

Conversely, suppose some n_j is composite. Its least prime factor q satisfies

$$q \le \sqrt{n_j} \le \sqrt{n_{k-1}} < s.$$

Since r, s are consecutive primes,

$$q \le r.$$

Thus

$$q \mid P_r$$

and

$$q \mid \prod_j n_j,$$

contradicting the assumption that the greatest common divisor is 1.

This proves the theorem.

19.1 A Long Chain May Cross a Square Boundary

A long chain may cross a prime square without any endpoint lying exactly on that square.

The square horizon of a chain should therefore be determined by its largest endpoint; one must not require the entire chain to lie inside a single open square window.

The finitely many boundary terms with

$$n_0 \le r$$

can be handled directly by their endpoint identities.

Thus every concrete finite chain has a complete and terminating finite certification.

Chapter 20 Single-Prime Forbidden Classes for Arithmetic Chains

Fix an odd prime q.

A chain is blocked at the q-layer exactly when there exists

$$j \in \{0, \dots, k-1\}$$

such that

$$2(a + jh) + 1 \equiv 0 \pmod{q}.$$

Let

$$u \equiv 2^{-1} \pmod{q}.$$

Then the forbidden starting classes are

$$\boxed{a \equiv -u - jh \pmod{q}, \qquad 0 \le j < k.} \tag{20.1}$$

Let

$$\rho_q(k, h)$$

denote the number of distinct residue classes among them.

Theorem 20.1 Exact Number of Forbidden Classes at a Single Prime Layer

$$\boxed{\rho_q(k, h) = \begin{cases} 1, & q \mid h, \\ \min(k, q), & q \nmid h. \end{cases}} \tag{20.2}$$

Proof

If

$$q \mid h,$$

then all residues

$$-jh$$

are equal modulo q, so only one starting class is forbidden.

If

$$q \nmid h,$$

then h is invertible in

$$\mathbb{Z}/q\mathbb{Z}.$$

Hence

$$j \mapsto -u - jh$$

is injective on the first

$$\min(k, q)$$

positions.

If

$$k \geq q,$$

then j runs through at least q consecutive integers and therefore covers all q residue classes. This proves the theorem.

Chapter 21 Chain Capacity at a Finite CRT Layer

Let S be a finite set of odd primes. Distinct prime moduli are coprime.

Therefore, by the Chinese remainder theorem, the total number of admissible starting residue classes for the entire chain, treated as one organized object, is

$$\boxed{\prod_{q\in S}\left(q-\rho_q(k,h)\right).} \tag{21.1}$$

This formula gives the exact local capacity at the finite layer.

It answers the question:

> Within the finite multiplicative layers already incorporated, is an equal-gap chain completely forbidden by the local structure?

If the product is zero, then at least one local prime layer covers every starting residue class.

If the product is positive, admissible residue classes remain at the finite layer.

Both conclusions are local structural statements only.

A genuine prime chain is completed only when those admissible positions ultimately acquire atomic identity.

Chapter 22 Intrinsic Step Capacity of Long Chains

Theorem 22.1 Necessary Step Condition for Long Chains

If

$$n_0 > k$$

and

$$C_k(a,h) = 1,$$

then

$$\boxed{\prod_{\substack{q \le k \\ q \text{ odd prime}}} q \mid h.} \tag{22.1}$$

Proof

Assume to the contrary that there exists an odd prime

$$q \le k$$

such that

$$q \nmid h.$$

By Theorem 20.1,

$$\rho_q(k,h) = q.$$

Thus the k endpoints cover every residue class modulo q, so at least one endpoint satisfies

$$n_j \equiv 0 \pmod{q}.$$

Since

$$n_j \ge n_0 > k \ge q,$$

we have

$$n_j > q$$

and $q \mid n_j$, so n_j is composite.

This contradicts

$$C_k(a,h) = 1.$$

This proves the theorem.

Corollary 22.2 A Fixed Step Cannot Support Unbounded Chain Length

Fix

$$h \ge 1.$$

Choose any odd prime q that does not divide h.

Then every equal-gap atomic chain whose endpoints are all larger than q must have length strictly less than q.

Hence arbitrarily long prime arithmetic progressions cannot be obtained by extending indefinitely with one fixed step h.

As chain length grows, the step must absorb progressively more low-order prime factors.

Chapter 23 Finite-Layer Nonemptiness Model

Fix k and a finite prime set S.

Choose h to be the product of all odd primes in S. Then for every

$$q \in S,$$

we have

$$q \mid h.$$

Consequently,

$$\rho_q(k,h) = 1.$$

The number of admissible classes remaining at the finite CRT layer is therefore

$$\boxed{\prod_{q\in S}(q-1) > 0.} \tag{23.1}$$

Thus

no fixed finite collection of low-order multiplicative layers automatically excludes an equal-gap chain with a suitably chosen step.

(23.2)

This is a finite-layer compatibility statement.

It does not replace local admissibility with the existence of a genuine atomic chain.

Chapter 24 Birth Law for Higher-Order Organizations

Let

$$a_1 < a_2 < \cdots$$

be all atoms, and write

$$A_t = \{a_1, \dots, a_t\}.$$

For $k \geq 2$, define

$$T_k(t) = \#\{(a,h): h \geq 1,\ C_k(a,h) = 1,\ a + (k-1)h \leq a_t\}.$$

Theorem 24.1 Birth Law for Higher-Order Organizations

For

$$t \geq 2,$$

we have

$$\boxed{T_k(t) - T_k(t-1) = \sum_{h=1}^{\left\lfloor \frac{a_t - 1}{k-1} \right\rfloor} \prod_{j=1}^{k-1} \alpha\,(a_t - jh).} \tag{24.1}$$

Proof

After the new atom a_t is added, every equal-gap chain whose largest endpoint is less than a_t has already been counted in $T_k(t-1)$.

Every newly added chain must therefore have

$$a_t$$

as its largest endpoint.

If its step is h, the remaining endpoints are uniquely

$$a_t - h,\ a_t - 2h, \dots, a_t - (k-1)h.$$

To ensure that the first endpoint is at least 1, one must have

$$h \leq \left\lfloor \frac{a_t - 1}{k-1} \right\rfloor.$$

The condition that all these historical endpoints are atoms is exactly that

$$\prod_{j=1}^{k-1} \alpha\,(a_t - jh)$$

equal 1.

Summing over all admissible h gives the formula.

This proves the theorem.

Thus higher-order relations do not suddenly appear independently of lower-order atoms.

Every k-endpoint chain is determined at the birth of its largest endpoint by a finite collection of historical endpoints.

Chapter 25 The Green–Tao Theorem in the Unified Relational Language

The Green–Tao theorem states that the primes contain arithmetic progressions of arbitrary length.

In atomic coordinates, it can be written as

$$\boxed{\forall k \geq 2,\ \forall M \geq 1,\ \exists a > M,\ \exists h \geq 1 : C_k(a,h) = 1.} \tag{25.1}$$

This says:

> For any prescribed length k, and no matter how far the lower threshold for the starting point is pushed, there exists a complete k-endpoint equal-gap atomic chain.

In ordinary prime coordinates, the corresponding common difference is $2h$.

The Green–Tao theorem itself supplies this global existence. What the present part establishes is its exact interface with the unified atomic relation matrix, finite horizons, relation birth, and fixed-sum folding.

25.1 Reference Theorem

Green–Tao Theorem. The primes contain arbitrarily long arithmetic progressions.

Reference:

Ben Green and Terence Tao, *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics **167** (2008), 481–547.

Chapter 26 Rigorous Transfer from Arithmetic Chains to High Goldbach Multiplicity

Theorem 18.1 together with the Green–Tao theorem yields two direct consequences.

Theorem 26.1 Unbounded Fixed-Sum Multiplicity

For every

$$R \geq 1,$$

there exist infinitely many even integers N such that

$$\boxed{G(N) \geq R.} \tag{26.1}$$

Proof

Take a prime arithmetic progression of length

$$2R.$$

By the mirror-folding theorem, its first-plus-last sum N has at least

$$R$$

distinct unordered prime representations.

The Green–Tao theorem permits such chains to occur arbitrarily far out, hence gives infinitely many distinct chains.

If these chains produced only finitely many values of N, then all such N would have a common upper bound; all endpoints of the chains would then also have a common upper bound, allowing only finitely many such chains, contrary to their occurrence arbitrarily far out.

This proves the theorem.

Theorem 26.2 High-Multiplicity Fixed Sums Centered at a Prime

For every

$$R \geq 1,$$

there exist infinitely many odd primes p such that

$$\boxed{G(2p) \geq R + 1.} \tag{26.2}$$

Proof

Take a prime arithmetic progression of length

$$2R + 1.$$

Its central term p is prime.

Mirror folding gives R prime representations from the pairs symmetric about the center; adding

$$p + p = 2p$$

gives at least

$$R + 1$$

unordered representations in total.

The Green–Tao theorem allows such chains to occur at arbitrarily large positions, hence gives infinitely many central primes. This proves the theorem.

These two theorems show that

arithmetic chains are not merely a third parallel readout;
they transmit arbitrarily high genuine relation multiplicities into fixed-sum slices.

(26.3)

However, “infinitely many fixed sums of high multiplicity” and “every admissible fixed sum is nonempty” are different quantified propositions.

This distinction is precisely why the quantifier layer must be established separately.

Chapter 27 A Five-Term Arithmetic-Progression Example

Consider the prime arithmetic progression

$$5,11,17,23,29.$$

Its atomic coordinates are

$$2,5,8,11,14.$$

Thus

$$k = 5, \qquad a = 2, \qquad h = 3,$$

and

$$C_5(2,3) = 1.$$

Mirror folding gives three representations of the same even integer

$$N = 34:$$

$$\boxed{5 + 29 = 11 + 23 = 17 + 17 = 34.} \tag{27.1}$$

Here

- the five primes form one complete equal-gap organization;
- 17 is the central prime;
- the same higher-order organization outputs three lower-order witnesses into a fixed-sum slice.

The higher-order structure is usable precisely because every output can be explicitly recovered to a concrete prime pair.

Part V Windows, Indices, Pullbacks, and Complete Factor Identities

Chapter 28 Window Relations

Not every relation is formed from finitely many fixed offset points.

Another large class of relations reads a finite interval.

Given a window

$$I_\theta \subset \mathbb{Z},$$

define the prime-occupancy count

$$\boxed{W(\theta) = \sum_{n \in I_\theta} \alpha\,(n).} \tag{28.1}$$

Window nonemptiness is equivalent to

$$\boxed{W(\theta) \geq 1.} \tag{28.2}$$

28.1 Legendre Windows

Define

$$\boxed{I_n = \{m : n^2 < m < (n+1)^2\}.} \tag{28.3}$$

The local relation is

$$\boxed{L(n) = \bigvee_{n^2 < m < (n+1)^2} \alpha(m).} \tag{28.4}$$

28.2 Oppermann Double Windows

Split the neighborhood of a square into

$$I_n^- = (n(n-1), n^2),$$

and

$$I_n^+ = (n^2, n(n+1)).$$

The local predicate requires

$$\boxed{W^-(n) \geq 1 \quad \text{and} \quad W^+(n) \geq 1.} \tag{28.5}$$

Thus Oppermann is not a single-window relation, but a pair of window relations sharing the square boundary.

28.3 Brocard Windows Between Consecutive Prime Squares

Let

$$p_n, p_{n+1}$$

be consecutive primes.

Define

$$\boxed{B(n) = \pi(p_{n+1}^2) - \pi(p_n^2).} \tag{28.6}$$

The Brocard-type local predicate is

$$\boxed{B(n) \geq 4.} \tag{28.7}$$

Here the parameter itself is generated by prime indexing, and the relation reads a window count between two consecutive prime squares.

Chapter 29 Relations Indexed by Consecutive Primes

Let

$$p_n$$

be the n-th prime.

Relations between consecutive primes need not read only a prime/composite Boolean value; they may read numerical functions generated by the fixed prime sequence.

29.1 Andrica-Type Relation

Define

$$\boxed{A_n = \sqrt{p_{n+1}} - \sqrt{p_n}.} \tag{29.1}$$

The local predicate is

$$\boxed{A_n < 1.} \tag{29.2}$$

Its outer structure is indexwise universal:

$$\forall n, \quad A_n < 1.$$

29.2 Firoozbakht-Type Relation

The original form is

$$p_{n+1}^{1/(n+1)} < p_n^{1/n}.$$

To avoid root-extraction errors in numerical computation, one may use the equivalent logarithmic form

$$\boxed{\frac{\log p_{n+1}}{n+1} < \frac{\log p_n}{n}.} \tag{29.3}$$

Thus this class of relations still comes from the fixed prime spectrum; only the local predicate has become a comparison of real numbers.

Chapter 30 Exponential and Polynomial Pullbacks

30.1 Mersenne-Type Exponential Pullback

Take a prime exponent p and define

$$\boxed{M(p) = 2^p - 1.} \tag{30.1}$$

The local relation is

$$\boxed{R_M(p) = \alpha(2^p - 1).} \tag{30.2}$$

If the question is whether there are infinitely many Mersenne primes, the outer quantifier is

$$\exists^{\infty} p \in \mathbb{P}: R_M(p) = 1.$$

This is the pullback of the fixed spectrum under an exponential map.

30.2 Bunyakovsky-Type Polynomial Pullback

Given an integer-coefficient polynomial

$$f(n),$$

define

$$\boxed{R_f(n) = \alpha(f(n)).} \tag{30.3}$$

The prime set on the integer spectrum is thereby pulled back to the input parameter axis n.

30.3 Schinzel H Polynomial Tuples

Given polynomials

$$f_1, \ldots, f_k,$$

define

$$\boxed{R_{\boldsymbol{f}}(n) = \prod_{j=1}^{k} \alpha\left(f_j(n)\right).} \tag{30.4}$$

This has the same formal structure as a fixed-offset pattern,

$$\prod_j \alpha\left(\text{position}_j\right),$$

except that the position maps have been upgraded from linear translations to general polynomials.

Chapter 31 Relations of Complete Factor Identity

The fixed spectrum contains fields richer than the prime/composite distinction.

Let

$$\mathrm{PF}(n)$$

denote the set of distinct prime factors of n.

This allows certain relations to read the "internal identity" of composite integers directly.

31.1 Grimm-Type Factor Matching

Consider a block of consecutive composite integers

$$n+1, \dots, n+k.$$

Each position carries a prime-factor set

$$\mathrm{PF}(n+i).$$

Construct a bipartite graph:

- left vertices: positions $1, \dots, k$;
- right vertices: prime factors that occur in the block;
- draw an edge between position i and prime p whenever $p \mid n+i$.

The relation predicate requires a matching that assigns to every position a distinct prime factor.

Thus the local object of a Grimm-type relation is

$$\boxed{\text{complete factor sets} + \text{bipartite matching.}} \tag{31.1}$$

31.2 Erdős–Woods-Type Identity Blocks

Take two starting points x, y and a block length k.

Compare

$$\mathrm{PF}(x+i), \qquad \mathrm{PF}(y+i),$$

for

$$0 \le i \le k.$$

The condition that the local identity blocks agree is

$$\boxed{\mathrm{PF}(x+i) = \mathrm{PF}(y+i) \quad \text{for every } i.} \tag{31.2}$$

Such relations go beyond the binary prime indicator, but still belong to the same fixed arithmetic identity matrix.

Part VI Quantifier Structures: Universality, Infinitude, and Other Global Requirements

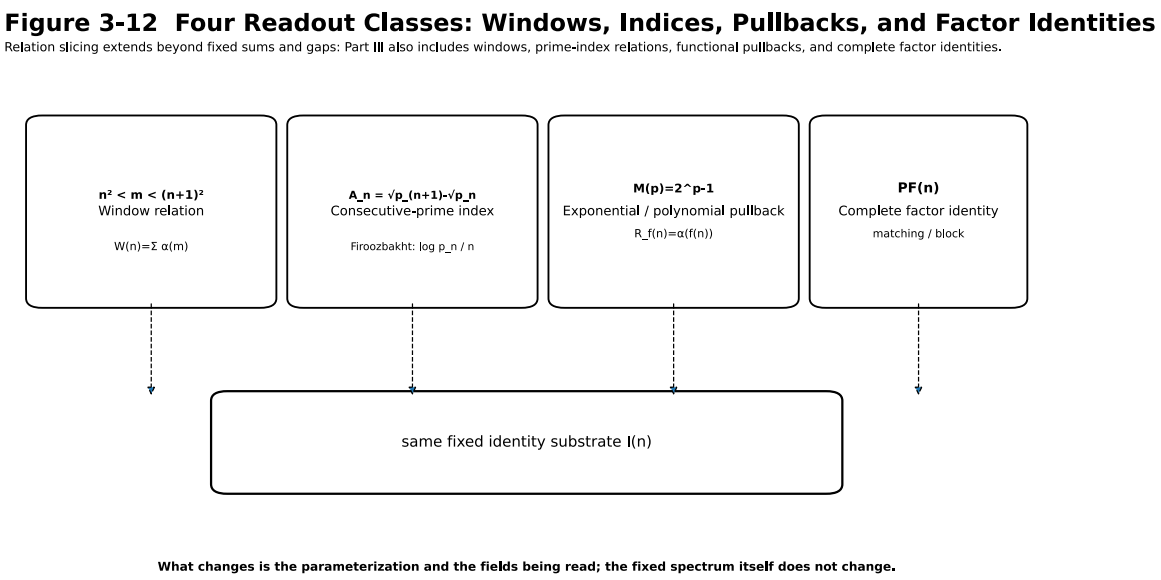


Figure 3-12. Four Readout Classes: Windows, Indices, Pullbacks, and Factor Identities. Part III extends relation slicing beyond fixed sums and gaps to four further classes: finite-window occupancy, functions indexed by consecutive primes, pullbacks of the fixed spectrum along exponential or polynomial maps, and complete prime-factor identities with matching relations. What changes is the parameterization and the field being read; the fixed arithmetic identity spectrum itself does not change.

Chapter 32 Quantifiers Are Not Decoration: They Determine the Global Type of a Proposition

Two problems can have exactly the same local relation yet become different mathematical propositions once their outer quantifiers differ.

Quantifiers must therefore be classified independently.

32.1 Finite Readout

Form:

$$\boxed{R_\Theta(x) \quad \text{is computed directly within finite parameters and a finite horizon.}} \tag{32.1}$$

This answers only a finite fact and does not add any global proposition.

32.2 Infinite Recurrence

Form:

$$\boxed{\exists^\infty x: R(x) = 1.} \tag{32.2}$$

Define

$$C_R(X) = \sum_{x \le X} R\,(x).$$

Then

$$\boxed{\exists^\infty x: R(x) = 1 \Leftrightarrow C_R(X) \text{ is unbounded.}} \tag{32.3}$$

This is the most basic infinitude criterion.

32.3 Parameterwise Universal Fibers

Form:

$$\boxed{\forall \theta \in \Theta, \quad \exists x: R_\theta(x) = 1.} \tag{32.4}$$

Define the fiber count

$$C(\theta) = \sum_{x} R_\theta\,(x).$$

Then

$$\boxed{\forall \theta\ \exists x: R_\theta(x) = 1 \Leftrightarrow C(\theta) > 0 \quad \forall \theta.} \tag{32.5}$$

Goldbach-type propositions belong to this class.

32.4 Window Universality

Form:

$$\boxed{\forall n, \quad \exists x \in I_n: \alpha(x) = 1.} \tag{32.6}$$

Define

$$W(n) = \sum_{x \in I_n} \alpha\,(x).$$

Then

$$\boxed{W(n) \geq 1 \quad \forall n} \tag{32.7}$$

is exactly equivalent to window universality.

Legendre and Oppermann belong to this type.

32.5 Arbitrary Length

Form:

$$\boxed{\forall k, \quad \exists \Theta, x \colon R_{k,\Theta}(x) = 1.} \tag{32.8}$$

Define the maximum realized length below horizon X by

$$L(X) = \max\{k \colon \exists \Theta, x \leq X, R_{k,\Theta}(x) = 1\}.$$

Then

$$\boxed{\forall k\ \exists \Theta, x \Leftrightarrow L(X) \text{ is unbounded.}} \tag{32.9}$$

Green–Tao belongs to this type.

32.6 Parameter Infinitude

Form:

$$\boxed{\exists^{\infty} \theta, \quad \exists x \colon R_{\theta}(x) = 1.} \tag{32.10}$$

This asks whether the successful parameters themselves are infinite in number, rather than whether a fixed parameter admits infinitely many anchors.

32.7 Indexwise Universal Predicates

Form:

$$\boxed{\forall n, \quad P(n).} \tag{32.11}$$

Examples include

- Andrica;
- Firoozbakht;
- Brocard;
- certain blockwise Grimm-type propositions.

Define the failure count

$$F(X) = \sum_{n \leq X} \mathbf{1}_{\{\neg P(n)\}}.$$

Universality is equivalent to

$$\boxed{F(X) = 0 \quad \forall X.} \tag{32.12}$$

32.8 Identity-Block Separation

This type compares finite identity blocks at two distinct starting points.

The local condition is

$$B_k(x) = B_k(y),$$

where

$$B_k(x) = (\mathrm{PF}(x), \ldots, \mathrm{PF}(x+k)).$$

A global separation proposition requires a uniform finite length beyond which distinct starting points must be distinguishable.

This quantifier structure is entirely different from ordinary infinite recurrence and must be treated separately.

Figure 3-13 Seven Global Quantifier Interfaces

The same relation matrix can be read by seven principal global quantifiers; each requires its own finite certificate.

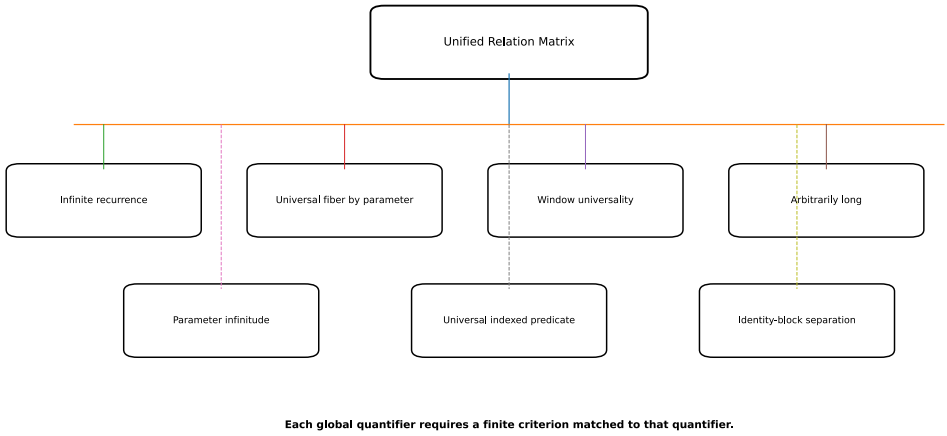


Figure 3-13. Seven Global Quantifier Interfaces. The same relation matrix admits seven principal global readouts: infinite recurrence, universal fibers by parameter, window universality, arbitrarily long structures, parameter infinitude, universal indexed predicates, and identity-block separation. Each quantifier requires a matching finite certificate.

Chapter 33 Why “Universal” and “Infinite” Must Be Kept Separate

The following two propositions both superficially involve infinitely many integers:

$$\forall N\ \exists p,$$

and

$$\exists^{\infty} p.$$

But the proof structures they demand are fundamentally different.

33.1 Universal Fibers Require “Every Column Nonempty”

If

$$C(N) = \sum_{p} R_N(p),$$

then one needs

$$\boxed{C(N) > 0 \quad \text{for every } N.} \tag{33.1}$$

No finite number of successful values of N, no positive average, and no unbounded cumulative count can replace this condition.

33.2 Infinite Recurrence Requires “The Total Number of Hits Keeps Growing”

If

$$C_R(X) = \sum_{x \le X} R(x),$$

then it suffices that

$$\boxed{C_R(X) \text{ is unbounded.}} \tag{33.2}$$

Arbitrarily large gaps are permitted here.

Therefore

universality means no parameter is missed;
infinitude means there is no final hit.

(33.3)

This distinction is the first discipline of batch proof.

Part VII Batch Proof and Global Closure

Chapter 34 Batch Proof: One Proof Covers an Entire Parameter Family

Checking one million parameters one by one still gives only one million finite facts.

The essence of batch proof is not “performing many computations” but having one uniform rule cover the entire parameter family.

Definition 34.1 Batch Certificate

Let the parameter space be

$$\Theta,$$

and let the target proposition be

$$\mathcal{P}(\theta).$$

Suppose there exists a certificate-state space

$$\mathcal{S}$$

and a map

$$\boxed{\mathcal{C}: \Theta \to \mathcal{S},} \tag{34.1}$$

satisfying the following two conditions.

Coverage

Every

$$\theta \in \Theta$$

enters some valid certificate state.

Correctness

Every valid certificate state implies

$$\mathcal{P}(\theta).$$

Then

$$\mathcal{C}$$

is called a batch certificate for the parameter family.

Consequently,

$$\boxed{\forall \theta \in \Theta, \quad \mathcal{P}(\theta).} \tag{34.2}$$

34.1 Batch Proof Is Not Batch Enumeration

Batch enumeration has the form

$$\theta_1, \theta_2, \dots, \theta_M$$

checked one by one.

Batch proof has the form

$$\boxed{\text{the same finite rule holds for arbitrary } \theta.} \tag{34.3}$$

Thus even an infinite parameter family can be controlled by a finite rule.

Chapter 35 Batch Decision for Universal Fibers

Let

$$R_\theta(x)$$

be a local relation.

Define the fiber

$$\mathcal{F}_\theta = \{x : R_\theta(x) = 1\}.$$

The universal target is

$$\forall \theta, \quad \mathcal{F}_\theta \neq \varnothing.$$

A batch proof can take several forms.

35.1 Uniform Positive Lower Bound

If one can construct

$$L(\theta)$$

such that

$$\boxed{|\mathcal{F}_\theta| \geq L(\theta) > 0} \tag{35.1}$$

for all parameters, universality follows immediately.

35.2 Exclusion of Failure States

Define the bad state of an empty fiber by

$$\mathcal{B} = \{\theta : \mathcal{F}_\theta = \varnothing\}.$$

If one proves

$$\boxed{\mathcal{B} = \varnothing,} \tag{35.2}$$

universality is likewise closed.

35.3 Exclusion of a Least Counterexample

If the parameter space is well ordered, for example the positive integers, assume there exists a least failing parameter

$$\theta_0.$$

If all smaller parameters satisfy the relation and the matrix recurrence then forces θ_0 to satisfy it as well, one obtains a contradiction.

This is a standard form of propagation-based batch proof.

Chapter 36 Batch Decision for Infinite Recurrence

The infinite-recurrence target is

$$\exists^{\infty} x: R(x) = 1.$$

By the counting criterion, it suffices to prove that

$$C_R(X)$$

is unbounded.

36.1 Direct Growth

If there exists a monotone unbounded function

$$L(X)$$

such that

$$\boxed{C_R(X) \geq L(X),} \tag{36.1}$$

then infinite recurrence follows.

36.2 Cumulative Net Potential

Suppose every finite block has relation supply

$$B_k$$

and maximal absorption capacity

$$C_k.$$

Define the net potential

$$\boxed{e_k = B_k - C_k.} \tag{36.2}$$

and its cumulative sum

$$\boxed{\mathcal{E}(N) = \sum_{k \leq N} e_k\,.} \tag{36.3}$$

If the cumulative number of genuine hits $T(N)$ satisfies

$$T(N) \geq \mathcal{E}(N),$$

and

$$\boxed{\limsup_{N \to \infty} \mathcal{E}(N) = +\infty,} \tag{36.4}$$

then

$$T(N)$$

is unbounded, and the relation recurs infinitely often.

This type of structure transforms infinitude into the question

whether long-term supply can be completely cancelled by long-term absorption.

(36.5)

Chapter 37 Four Basic Modes of Closure

Common closure mechanisms in batch proofs can be grouped into four basic types, together with one especially important finite-to-global assembly principle.

37.1 Finite-Quotient Closure

Suppose an infinite parameter family has only finitely many states under an equivalence relation:

$$\Theta/\sim = \{s_1, \dots, s_M\},$$

and the target depends only on the state. Then checking these finitely many states covers all parameters.

Finite modular structures and CRT constructions often yield this type of closure.

37.2 Propagation Closure

Suppose there is a rule

$$\boxed{P(\theta) \Rightarrow P(T\theta)} \tag{37.1}$$

and that the orbits of finitely many starting points under T cover the parameter space. Then a finite base together with the propagation rule closes the universal statement.

37.3 Cumulative-Potential Closure

Suppose a cumulative relation quantity admits a uniform lower bound

$$C(X) \geq E(X),$$

with

$$E(X)$$

unbounded. Then infinite recurrence is closed.

37.4 Bad-State Exclusion Closure

Suppose all failure states form

$$\mathcal{F}_{\text{bad}},$$

all reachable states form

$$\mathcal{R},$$

and one proves

$$\boxed{\mathcal{F}_{\text{bad}} \cap \mathcal{R} = \varnothing.} \tag{37.2}$$

Then failure cannot occur.

37.5 Local-Finite / Global-Assembly Closure

Universal propositions admit another especially important form of closure:

An infinite target space can be partitioned into a sequence of disjoint finite terminal segments, and each target needs to be certified only inside the unique segment to which it belongs.

Suppose the target space Y decomposes as

$$\boxed{Y = \bigsqcup_{k\geq 1} J_k.} \tag{37.3}$$

Let the persistent relation system satisfy

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_\infty,$$

and let the relation coordinate be

$$\theta: \Gamma_\infty \to Y.$$

Define the support of the k-th layer readout by

$$\boxed{S_k^\theta = \theta(\Gamma_k).} \tag{37.4}$$

Assume every target $y \in J_k$ is already terminal at layer k:

$$\boxed{y \in J_k \Rightarrow \Gamma_\infty^{(\theta=y)} = \Gamma_k^{(\theta=y)}.} \tag{37.5}$$

Then we have the following.

Theorem 37.1 Local-Finite / Global-Assembly Theorem

Under the preceding assumptions,

$$\boxed{\forall y \in Y, \quad \Gamma_\infty^{(\theta=y)} \neq \varnothing} \tag{37.6}$$

if and only if

$$\boxed{\forall k, \quad J_k \subseteq S_k^\theta.} \tag{37.7}$$

Proof

First assume

$$\forall y \in Y, \quad \Gamma_\infty^{(\theta=y)} \neq \varnothing.$$

Fix k and $y \in J_k$.

By terminality,

$$\Gamma_\infty^{(\theta=y)} = \Gamma_k^{(\theta=y)}.$$

The left-hand side is nonempty, so the right-hand side is nonempty; hence

$$y \in S_k^\theta.$$

Therefore

$$J_k \subseteq S_k^\theta.$$

Conversely, assume that for every k,

$$J_k \subseteq S_k^\theta.$$

Fix any $y \in Y$.

Because

$$Y = \bigsqcup_{k\geq 1} J_k,$$

there is a unique k such that

$$y \in J_k.$$

Then

$$y \in S_k^{\theta},$$

so

$$\Gamma_k^{(\theta=y)} \neq \varnothing.$$

Since

$$\Gamma_k \subseteq \Gamma_\infty,$$

it follows that

$$\Gamma_\infty^{(\theta=y)} \neq \varnothing.$$

This proves the theorem.

Therefore

an infinite universal quantifier is losslessly rewritten as infinitely many finite terminal propositions.

(37.8)

The infinite target space has not been "computed to completion."

What actually happens is

each target → its unique finite terminal segment → a finite nonemptiness decision in that segment.

The natural terminal segments for fixed sums are the canonical example of this master theorem; Part IV develops them concretely.

Part VIII Relation Equivalence, Executable Syntax, and Master Theorems

Chapter 38 Relation Equivalence and Relation Difference

The unified language further clarifies the question:

> When are two differently written relations merely different coordinate descriptions of the same relation?

Definition 38.1 Lossless Relation Equivalence

Let two relation descriptions

$$R \quad \text{and} \quad R'$$

have relation-instance spaces

$$\mathcal{X}, \qquad \mathcal{X}'.$$

If there exists a bijection

$$\Phi: \mathcal{X} \to \mathcal{X}'$$

such that

1. endpoint identities in the fixed spectrum correspond bijectively;
2. truth values of the local predicates are preserved;
3. the outer quantifiers are preserved under the map;

then the two relations are called **losslessly relation-equivalent**.

38.1 Goldbach in Atomic Coordinates and Ordinary Prime Coordinates

Ordinary coordinates:

$$p + q = N.$$

Atomic coordinates:

$$a + b + 1 = m, \qquad N = 2m.$$

The correspondence

$$p = 2a + 1, \qquad q = 2b + 1$$

is bijective.

Thus the two forms are merely lossless coordinate transformations of the same relation.

38.2 Fixed Difference and Horizontal Slices of the Relation Plane

Ordinary coordinates:

$$q - p = 2h.$$

Relation-plane coordinates:

$$(m, h) \in \Gamma.$$

These are also losslessly equivalent.

38.3 Equal Counts Do Not Imply Relation Equivalence

If two relation states satisfy only

$$|R_1| = |R_2|,$$

it does not follow that they are relation-equivalent.

The relation-sieve example in Part II already showed that the same number of surviving slots can lead to different updates at the next layer.

Therefore

equal numerical values do not imply equal relations.

(38.1)

Chapter 39 Minimal Operations of the Relational Language

The unified relational language requires only a small collection of basic operations.

R1 Position Mapping

$$x \mapsto f_{\Theta}(x).$$

R2 Identity Readout

$$f_{\Theta}(x) \mapsto \boldsymbol{I}(f_{\Theta}(x)).$$

R3 Finite Logic and Finite Comparisons

Including

$$\wedge, \quad \vee, \quad \neg,$$

and

$$=, \quad <, \quad >, \quad \#, \quad \text{matching}.$$

R4 Window Aggregation

For example,

$$\sum_{n \in I} \alpha\,(n),$$

or

$$\prod_{n \in I} (1 - \alpha(n)).$$

R5 Outer Quantifiers

Including

$$\forall, \qquad \exists, \qquad \exists^{\infty},$$

and nested quantifiers carrying length, window, or block parameters.

Thus

the relational language does not need to redefine primality;
it needs only to define positions, readouts, combinations, and quantifiers.

(39.1)

Chapter 40 From Mathematical Syntax to an Executable Relational Language

To write the relation normal form in executable form, it is enough to make the following fields explicit:

anchor	anchor variable and its finite readout range,
parameters	relation parameters,
maps	family of position maps,
predicate	local relation predicate,
quantifier	outer quantifier,
certificate	optional finite-certificate module.

(40.1)

This is the mathematical basis for a general relation-description language.

Predefined names such as Goldbach, Twin, and Legendre are only frequently used templates.

The true boundary of the theory is determined by the relational syntax, not by the conjectures already listed in a menu.

Therefore

the predefined catalogue is not the boundary of the theory;

relational syntax is the theoretical interface.

(40.2)

Chapter 41 Three Custom Relation Examples

The unified syntax not only reconstructs existing problems; it can also generate new relation slices naturally.

41.1 Cross-Mirror Relation

Require

$$p, \qquad p+2, \qquad 2p+1$$

to be prime simultaneously.

Define

$$\boxed{X(p) = \alpha(p)\alpha(p+2)\alpha(2p+1).} \tag{41.1}$$

This couples a fixed translation with an affine map.

If the question is whether it occurs infinitely often, the outer quantifier is

$$\exists^{\infty} p : X(p) = 1.$$

41.2 Quadratic Double-Atomic Relation

Define

$$f_1(n) = n^2 + n + 41,$$

$$f_2(n) = n^2 + n + 43.$$

The local relation is

$$\boxed{Q(n) = \alpha(f_1(n))\alpha(f_2(n)).} \tag{41.2}$$

This is a double-polynomial pullback.

41.3 Reflection–Translation Coupled Relation

Fix an even integer N, and require

$$p, \qquad p+2, \qquad N-p$$

to be prime simultaneously.

Define

$$\boxed{H_N(p) = \alpha(p)\alpha(p+2)\alpha(N-p).} \tag{41.3}$$

This uses simultaneously

- a fixed difference;
- a fixed-sum reflection.

If such a p is required to exist for every admissible N, the proposition is of the parameterwise universal-fiber type.

These examples show that

the unified relational language not only organizes existing problems;
it can systematically generate new relation objects.

(41.4)

Chapter 42 Relation Type and Quantifier Type as Two Independent Coordinates

The relation type answers:

> *What does the local object look like?*

The quantifier type answers:

> *Over how large a parameter family is this local object required to hold?*

For example, the fixed pattern

$$H = \{0,2\}$$

is itself only a local relation.

One can ask whether

$$R_H(n) = 1$$

holds for one given n;

or ask

$$\exists n: R_H(n) = 1;$$

or ask

$$\exists^{\infty} n: R_H(n) = 1.$$

All three use exactly the same local relation but define three different propositions.

Therefore

relation type and quantifier type must be preserved orthogonally.

(42.1)

This is one reason the unified language avoids conflating distinct problems.

Chapter 43 Unified Table of Classical Relations

Relation	Position map / relation object	Local predicate	Typical quantifier
Goldbach	$p, N-p$	both endpoints prime	$\forall N \exists p$
Lemoine/Lévy	$p, (N-p)/2$	both endpoints prime	$\forall N \exists p$
Twin	$p, p+2$	both endpoints prime	$\exists^{\infty} p$
fixed gap	$p, p+d$	both endpoints prime	depends on proposition
Polignac	$p, p+d$ + interior window	prime endpoints and no interior prime	$\exists^{\infty} p$
Sophie Germain	$p, 2p+1$	both endpoints prime	$\exists^{\infty} p$
prime tuple	$n+H$	all endpoints prime	$\exists^{\infty} n$
Green–Tao	$a, a+h, \dots, a+(k-1)h$	all endpoints atomic	$\forall k \exists a, h$
Legendre	$(n^2, (n+1)^2)$	window nonempty	$\forall n$
Oppermann	two half-square windows	both windows nonempty	$\forall n$
Brocard	(p_n^2, p_{n+1}^2)	at least 4 primes in the window	$\forall n$
Andrica	p_n, p_{n+1}	square-root difference less than 1	$\forall n$
Firoozbakht	p_n, p_{n+1}	logarithmic root monotonicity	$\forall n$
Mersenne	$p \mapsto 2^p - 1$	image value prime	$\exists^{\infty} p$
n^2+1	$n \mapsto n^2+1$	image value prime	$\exists^{\infty} n$
Bunyakovsky	$n \mapsto f(n)$	image value prime	$\exists^{\infty} n$
Schinzel H	$n \mapsto (f_1(n), \dots, f_r(n))$	all values prime	$\exists^{\infty} n$
Grimm	consecutive composite block	matching by distinct prime factors	blockwise universality
Erdős–Woods	two factor-identity blocks	compare block identities	identity-block separation

This is not a summary table of conjectures. It is a classification table for the relational language.

It shows that problems that appear very different can fall into a finite collection of underlying readout types.

Chapter 44 Minimal Completeness Conditions for Batch Proof

A genuine batch proof covering an infinite parameter family must satisfy at least four conditions simultaneously.

44.1 Relation Fidelity

The certificate must preserve the relation information required by the target proposition.

If the target requires reflection pairings, marginal density alone is insufficient.

If the target requires complete factor sets, the least prime factor alone is insufficient.

44.2 Finite Local Correctness

For every fixed parameter, the finite certificate decision must agree with the truth value of the original relation.

44.3 Parameter Coverage

The certificate must apply to all target parameters, not merely to most parameters, average parameters, or a finite checked range.

44.4 Quantifier Closure

The certificate must match the target quantifier:

- universality requires every parameter fiber to be nonempty or every failure state to be unreachable;
- infinite recurrence requires an unbounded cumulative number of hits;
- arbitrary length requires the maximum realized length to be unbounded;
- identity-block separation requires a uniform finite separating length.

Therefore

$$\boxed{\text{relation fidelity} + \text{local correctness} + \text{parameter coverage} + \text{quantifier closure} = \text{the complete s}} \tag{44.1}$$

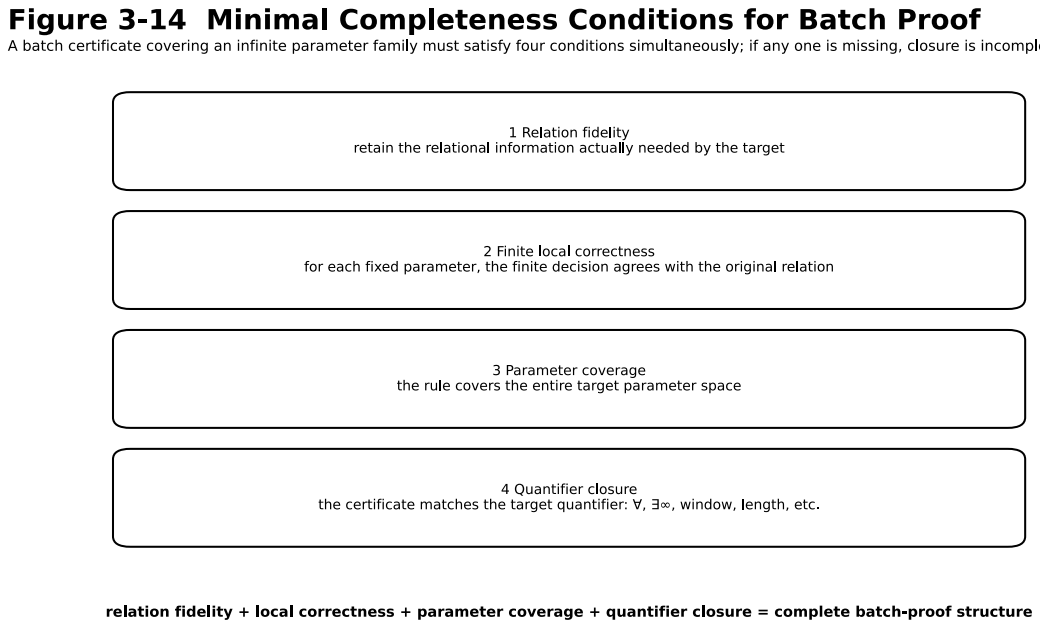


Figure 3-14. Minimal Completeness Conditions for Batch Proof. A batch proof over an infinite parameter family must simultaneously satisfy four conditions: relation fidelity, finite local correctness, parameter coverage, and quantifier closure. Relevant relational information may not be lost under compression; the finite decision for each fixed parameter must be correct; the rule must cover the full target parameter space; and the final certificate must match the outer quantifier, such as universality, infinite recurrence, window universality, or arbitrary length.

Chapter 45 When Higher-Order Structure Is Genuinely Useful

A higher-order structure is not automatically stronger merely because it is “higher order.”

It becomes genuinely useful only when it satisfies

$$\boxed{\text{higher-order structure} \rightarrow \text{recoverable lower-order witnesses.}} \tag{45.1}$$

For example:

- long Green–Tao chains can be folded into concrete Goldbach prime pairs;
- multi-column overlaps of fixed-sum witnesses can exactly recover prime constellations;
- the binary atomic sum–difference plane can losslessly recover concrete prime pairs.

These are legitimate uses of higher-order structure.

By contrast, if only an average remains and one can no longer determine whether a target pair of endpoints exists, then that average cannot independently complete a proof of the lower-order relation.

The book therefore preserves the basic discipline

structures may be lifted to higher order; witnesses must not be lost.

(45.2)

Chapter 46 Table of the Fundamental Theorems of This Part

Fundamental Theorem O Losslessness of the Binary Atomic Relation Plane

$$\boxed{(m,h) \leftrightarrow \{m-h, m+h\}.}$$

Fixed sum and fixed difference are different coordinate fibers of the same binary atomic relation object.

Fundamental Theorem O1 Fixed-Product Hyperbolic Leaves

$$\boxed{(m-h)(m+h) = m^2 - h^2.}$$

Thus fixed product M corresponds in the relation plane to

$$\boxed{m^2 - h^2 = M.}$$

Fundamental Theorem O2 Total Multiplicative-Depth Identity

$$\boxed{\Omega(m-h) + \Omega(m+h) = \Omega(m^2 - h^2).}$$

The total multiplicative depth on a fixed-product leaf equals $\Omega(M)$.

Fundamental Theorem O3 Shell-Intersection Bijection

$$\boxed{L_x^- \cap L_y^+ = \left\{\left(\frac{x+y}{2}, \frac{y-x}{2}\right)\right\}.}$$

Binary relation instances are in one-to-one correspondence with unique intersections of oppositely oriented shells.

Fundamental Theorem O4 Equivalence of Shell Growth and Relation Growth

If K atoms are already present and M new atoms are added, then the number of new unordered relations is

$$\boxed{KM + \frac{M(M+1)}{2}.}$$

Geometrically, this is exactly the old–new rectangle plus the new–new triangular region.

Fundamental Theorem O5 Orthogonal Generation

$$\boxed{C(x,y) = \frac{x}{2}(1,-1) + \frac{y}{2}(1,1),}$$

with

$$\boxed{(1,-1) \cdot (1,1) = 0.}$$

Prime gaps therefore become local geometric steps of the relation grid.

Fundamental Theorem P Witness-Persistence Identity

$$\boxed{\sum_{m} |\, W_K(m) \cap W_K(m+h)| = K\, T_{K,h}.}$$

Twin is the special case $h = 1$.

Fundamental Theorem Q Multi-Column Persistence Identity

$$\boxed{\sum_{m} \left| \bigcap_{h \in H} W_K(m+h) \right| = K\, T_K(H).}$$

Finite prime constellations can be read as persistent multi-column overlaps of the fixed-sum witness field.

Fundamental Theorem R Equal-Gap Chain Concatenation

$$\boxed{C_k(a,h) = \prod_{j=0}^{k-2} E_h\,(a + jh).}$$

Fundamental Theorem S Mirror Folding of Arithmetic Chains

$$\boxed{p_j + p_{k-1-j} = p_0 + p_{k-1}.}$$

Therefore

$$\boxed{G(N) \geq \left\lceil \frac{k}{2} \right\rceil.}$$

Fundamental Theorem T Number of Forbidden Classes at One Prime Layer

$$\boxed{\rho_q(k,h) = \begin{cases} 1, & q \mid h, \\ \min(k,q), & q \nmid h. \end{cases}}$$

Fundamental Theorem U Finite CRT Chain Capacity

$$\boxed{\prod_{q \in S} (\,q - \rho_q(k,h)).}$$

Fundamental Theorem V Necessary Step Condition for Long Chains

If the first endpoint of a complete k-endpoint equal-gap prime chain is larger than k, then

$$\boxed{\prod_{\substack{q \leq k \\ q \text{ odd prime}}} q \mid h.}$$

Fundamental Theorem W Birth Law for Higher-Order Organizations

$$\boxed{T_k(t) - T_k(t-1) = \sum_{h} \prod_{j=1}^{k-1} \alpha\,(a_t - jh),}$$

where h ranges over all integers making the first endpoint positive.

Fundamental Theorem X Infinite-Recurrence Criterion

$$\boxed{\exists^{\infty} x \colon R(x) = 1 \Leftrightarrow \sum_{x \le X} R\,(x) \text{ is unbounded.}}$$

Fundamental Theorem Y Universal-Fiber Criterion

$$\boxed{\forall \theta \exists x \colon R_\theta(x) = 1 \Leftrightarrow \sum_{x} R_\theta\,(x) > 0 \quad \forall \theta.}$$

Fundamental Theorem Z Batch-Certificate Principle

If a uniform certificate map simultaneously satisfies relation fidelity, local correctness, parameter coverage, and quantifier closure, then it covers the entire parameter family in one proof.

Fundamental Theorem Z1 Local-Finite / Global-Assembly Theorem

If

$$Y = \bigsqcup_{k \ge 1} J_k$$

and

$$y \in J_k \Rightarrow \Gamma_{\infty}^{(\theta=y)} = \Gamma_{k}^{(\theta=y)},$$

then

$$\boxed{\forall y \in Y,\ \Gamma_{\infty}^{(\theta=y)} \neq \varnothing \Leftrightarrow \forall k,\ J_k \subseteq S_k^{\theta}.}$$

Figure 3-15 Unified Relation Map of Part III

The relation plane is only the entry point; Part III culminates in a full compilation chain: relation objects → general language → quantifier types → batch certificates.

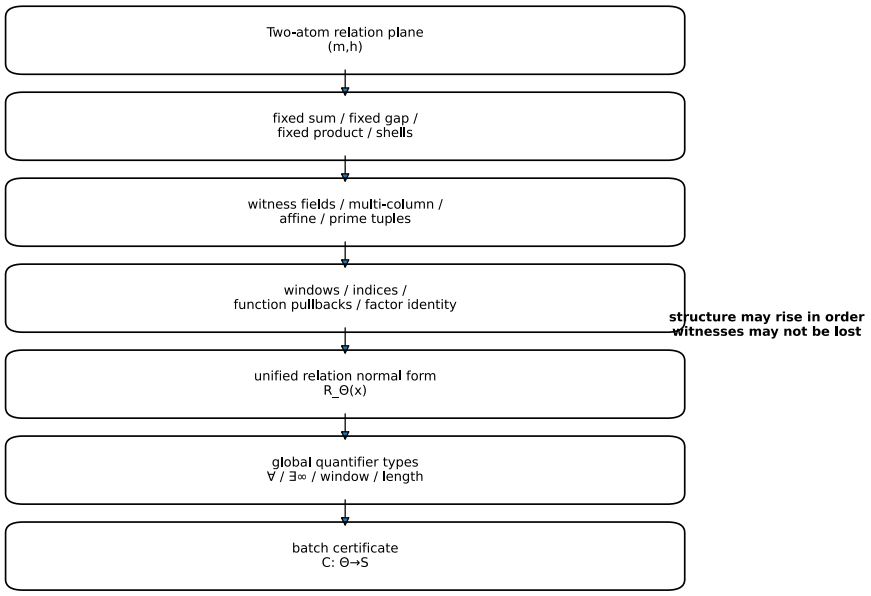


Figure 3-15. Unified Relation Map of Part III. Part III starts from the lossless two-atom relation plane, passes through fixed sums, fixed gaps, fixed products, shells, witness persistence, affine relations, and higher-order templates, then extends to windows, indices, functional pullbacks, and complete factor identities, finally compiling them into a unified relation normal form, a classification of global quantifiers, and batch certificates.

Conclusion One Spectrum Does Not Mean One Relation

The unification provided by a fixed spectrum can easily be misread as the overly simple claim:

> *Since prime positions are already fixed, relations are merely table lookups.*

This part establishes precisely the opposite conclusion.

Fixed prime identity only means that a relation does not need to manufacture primality anew.

It does not tell us

- which positions must be paired;
- which positions must appear simultaneously;
- which windows must be nonempty;
- which relations must be consecutive;
- which parameters carry universal requirements;
- which hits must recur infinitely often;
- which higher-order organizations must preserve lower-order witnesses.

All of these belong to the relation itself.

Therefore

the fixed spectrum answers "what is it?";
relation theory answers "how do they occur together?"

Sums, differences, affine maps, windows, polynomials, exponentials, adjacent primes, and complete factor sets are not redundant decorations of the fixed spectrum.

They are the ways in which fixed facts are organized into mathematical propositions.

Only because these relations are reconstructed at a sufficiently low level can genuine unification occur.

Goldbach and Twin are not merely "one reads sums and the other reads differences."

They inhabit the same binary atomic relation plane.

Nor does this plane carry only sum and difference.

From

$$(m-h)(m+h) = m^2 - h^2,$$

the same relation point also lies on a fixed-product hyperbolic leaf.

Thus

sum, difference, and product are three foliations of the same binary endpoint relation.

The endpoints themselves can also be written as two families of oppositely oriented shells,

$$L_x^-, \qquad L_y^+.$$

Every binary relation instance is exactly the unique intersection of two such shells, and the two endpoints generate the entire relation grid along the orthogonal directions

$$(1,-1), \qquad (1,1).$$

Prime gaps therefore acquire a second meaning:

they are both gaps in the one-dimensional atomic sequence
and local steps in two-dimensional relation geometry.

These geometric structures do not introduce a second collection of prime objects.

They merely unfold, without loss, the endpoint relations already present in the same fixed identity spectrum.

Twin can even be read exactly from persistence across adjacent columns of the fixed-sum witness field:

$$\boxed{\sum_m |W_K(m) \cap W_K(m+1)| = K\,T_{K,1}.}$$

More generally,

$$\boxed{\sum_m \left|\bigcap_{h\in H} W_K(m+h)\right| = K\,T_K(H).}$$

Hence

the same fixed-sum relation field carries exact finite readouts of all finite prime constellations.

Green–Tao, in turn, shows that higher-order arithmetic chains arise naturally in the same relation matrix and, through mirror folding, transmit arbitrarily high multiplicities into fixed-sum slices:

higher-order relations can enrich lower-order relations,
but they must remain recoverable to lower-order witnesses.

Universality and infinitude likewise cannot be blurred into one vague "infinite problem."

When the target space has a natural partition into finite terminal segments,

$$Y = \bigsqcup_{k\geq 1} J_k,$$

with each $y \in J_k$ terminal at layer k, then

an infinite universal quantifier can be losslessly rewritten as infinitely many finite terminal propositions.

This does not weaken universality into finite checking. It sends each target exactly back to the finite world to which it belongs.

Different quantifier types require, respectively,

every fiber nonempty,

unbounded cumulative hits,

every window occupied,

unbounded maximum realized length,

or another criterion exactly matched to the relevant quantifier.

The unified relation structure of the book can therefore be written as

fixed arithmetic identity spectrum
⇓
binary atomic relation geometry
⇓
lower-order relation witnesses
⇓
relation position maps
⇓
local relation predicates
⇓
relation slices
⇓
quantifier types
⇓
batch certificates.

This chain gives "unification" a precise meaning:

what is unified is the language, the matrix, and the proof interface;
what is preserved is the irreducible structure of each relation.

Only under this discipline does batch treatment of prime relations cease to be a forced identification of different problems and become a way for different problems to retain their full identity within one common foundational structure.

Part IV Finite-Control Geometry, Local Obstruction, and Relation Sieves

— From Local Phases, Survival Trees, and Genuine Death to the Terminal-Saturation Structure of Fixed Sums

Central theme of this part

The first three parts established

fixed identity → relation reconstruction → finite terminality → relation slices → quantifier structure.

This part enters the finite-control layer of relation theory.

The fixed spectrum tells us what every position ultimately is; relation slices tell us which positions must be viewed together; finite-control geometry asks the next question:

How does a finite multiplicative layer act on an entire relation without losing the relation itself?

Three kinds of structure must be preserved simultaneously:

multiplicative depth + relation position + genuine-death history.

This gives rise to a **relation sieve**:

an element sieve decides endpoint identities;
a relation sieve decides how those identities coexist, are jointly eliminated, or become terminal within a specified relation.

Fixed-sum relations provide the most complete terminal-saturation example in this part. They are not treated as an isolated technique, but as the concrete unfolding of general finite-control geometry on binary atomic reflection fibers; within this structure the universal closure of Goldbach is completed.

Abstract

This part establishes the finite-control geometry of Unified Prime Relations Theory.

Let the positive odd-number relation space be

$$\mathcal{X}_r = \mathbb{O}_{\geq 3}^{r}.$$

For a relation point

$$(x_1, \dots, x_r),$$

define the multiplicative-depth tensor

$$\Delta_r(x_1, \dots, x_r) = (\Omega(x_1), \dots, \Omega(x_r)).$$

All endpoints are odd primes exactly when

$$\Delta_r = (1, \dots, 1).$$

Accordingly, define the atomic vertex skeleton

$$\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1) = \mathbb{P}_{\mathrm{odd}}^{r}.$$

Any finite-order all-prime relation can be written as the intersection of a relation orbit

$$F: \mathcal{U} \to \mathcal{X}_r$$

with the atomic vertex skeleton:

$$\boxed{\{t: F_1(t), \dots, F_r(t) \text{ are all prime}\} = F^{-1}(\mathcal{V}_r).}$$

To describe a finite control layer, for each prime ℓ define the local obstruction set

$$B_\ell(F) = \left\{ t \bmod \ell : \prod_{j=1}^{r} F_j(t) \equiv 0 \pmod{\ell} \right\}.$$

If some $B_\ell(F)$ covers every parameter residue class, the relation is completely blocked by one local multiplicative layer. If no prime layer completely blocks it, the relation is said to satisfy local admissibility. Local admissibility excludes only total local obstruction; it does not by itself imply global occupancy.

For a finite offset pattern

$$H = \{h_1, \dots, h_k\},$$

define

$$\nu_H(q) = \#\{h \bmod q : h \in H\}, \qquad v_H(q) = q - \nu_H(q).$$

If

$$\nu_H(q) = q,$$

then every translate has at least one endpoint divisible by q, so—apart from finitely many boundary exceptions—the all-prime relation is completely blocked. If

$$\nu_H(q) < q$$

for every prime q, the pattern is locally admissible. To detect complete coverage, it is enough to check

$$q \le |H|.$$

The local capacity of finitely many layers can be developed further into an exact **CRT survival tree**. Let the control primes be introduced successively as

$$p_1, \dots, p_s, \qquad W_s = \prod_{i=1}^{s} p_i.$$

Define the joint surviving residue set $A_s(H)$ at layer s. Then

$$\boxed{|A_s(H)| = \prod_{i=1}^{s} (p_i - \nu_H(p_i)).}$$

If

$$\eta_i = 1 - \frac{\nu_H(p_i)}{p_i},$$

then the survival proportion is

$$\boxed{\frac{|A_s(H)|}{W_s} = \prod_{i=1}^{s} \eta_i .}$$

More strongly, all parameter phases modulo W_s split exactly into "joint final survival" and "first local elimination at some layer":

$$\boxed{1 = \prod_{i=1}^{s} \eta_i + \sum_{i=1}^{s} (1 - \eta_i) \prod_{j<i} \eta_j .}$$

Thus local sieving is not merely a final product. It carries a complete first-elimination history:

$$\boxed{\text{all parameter phases} = \text{joint survivors} \sqcup \text{first local eliminations at each layer.}}$$

This phase-level conservation law is then connected to genuine death, least prime factors, and the relation first killer.

For the arithmetic pattern

$$H_{k,d} = \{0, d, \dots, (k-1)d\},$$

local admissibility at every $q \le k$ requires each such prime to divide d.

At the level of individual identities, this part introduces a genuine-death partition. For an odd composite n, the unique first responsible prime is

$$P^-(n).$$

Define

$$D_q = \{n : P^-(n) = q\}.$$

The sets D_q are pairwise disjoint. Every odd composite therefore genuinely dies exactly once, at the layer of its least prime factor.

This principle is lifted to the relation level. A finite relation slot genuinely dies only when one of its endpoints is first certified composite, and it records a unique first killer. The state of a relation sieve is not merely "how many remain"; it must preserve concrete relation positions, endpoint identities, first killers, and the necessary relation history.

To write different terminal states as finite objects in a unified way, this part defines a **relation alphabet**. If terminal endpoint identities are denoted

$$A, \qquad \mathcal{M}, \qquad \partial,$$

then an r-endpoint relation has an identity word

$$\omega(R) = (\iota(n_1), \dots, \iota(n_r)).$$

The collection of all terminal identity words available to a relation class is

$$\boxed{\Sigma_R.}$$

An unordered binary fixed-sum slot has three coarse terminal words:

$$\boxed{AA, \quad A\mathcal{M}, \quad \mathcal{M}\mathcal{M}.}$$

In the old-double-escape purification layer of a single square horizon for Twin, because the same current control prime cannot divide both endpoints separated by 2, the within-layer alphabet contracts to

$$\boxed{AA, \quad \mathcal{M}_r A, \quad A\mathcal{M}_r.}$$

The relation alphabet turns "what a relation may ultimately become" into a finite, enumerable, and verifiable object.

A fixed-sum example gives an important counterexample: two targets can have the same number of initial slots and the same number of survivors after $q = 3$, yet undergo different updates at the $q = 5$ layer. Thus

$$(H, |S_q|)$$

is not a sufficient description of the relation state. Relation positions and residue configurations carry genuine information.

At the parity level of multiplicative depth, define

$$\lambda(n) = (-1)^{\Omega(n)}.$$

Genuine death at the least prime factor obeys

$$\lambda(qm) = -\lambda(m).$$

Hence the parity channels have an exact cross-transition. If all prime factors up to z have been processed and

$$z \geq X^{1/3},$$

then every genuine survivor in the odd multiplicative-depth channel within $[3, X]$ must be prime. This gives cube-root certification of the odd channel. At the shallower level

$$z \geq X^{1/4},$$

a genuine odd-channel survivor can have multiplicative depth only 1 or 3.

These shallower certification levels do not by themselves provide a positive lower bound for a fixed-sum fiber, because fixed sums require two-point reflection correlation:

$$\boxed{R_f(N) = \sum_x f(x) f(N - x) = \langle f, J_N f \rangle.}$$

Finite-modulus marginal data, total counts, or simple parity proportions do not in general recover this reflection pairing. The value of a relation sieve is therefore not to flatten the relation further, but to preserve the reflection phase exactly.

The second half of this part develops the fixed-sum relation completely as a terminal-saturation relation sieve. For the atomic sequence

$$A = \{a_1 < a_2 < \cdots\},$$

define the finite relation matrix

$$\Gamma_k = \{\{a_i, a_j\}: 1 \le i \le j \le k\},$$

the fixed-sum support

$$S_k = A_k \boxplus A_k,$$

and the natural terminal segment

$$\boxed{J_k = \{a_k + 2, \dots, a_{k+1} + 1\}.}$$

Then

$$\mathbb{Z}_{\ge 3} = \bigsqcup_{k \ge 1} J_k.$$

If

$$m \in J_k,$$

then no future atom can participate in a representation of sum m, so

$$\boxed{\Gamma_\infty^{(\sigma=m)} = \Gamma_k^{(\sigma=m)}.}$$

Fixed sums therefore acquire finite terminality.

For each terminal target m, define all unordered relation slots

$$\mathcal{S}_m = \{\{a, b\}: a + b + 1 = m,\ 1 \le a \le b\}.$$

After terminality, each endpoint has only the identities atom A or multiplicatively occupied $\mathcal{M}$. Hence

$$\boxed{\mathcal{S}_m = \mathcal{S}_m^{AA} \sqcup \mathcal{S}_m^{A\mathcal{M}} \sqcup \mathcal{S}_m^{\mathcal{M}\mathcal{M}}.}$$

Define

$$\hat{\alpha}(a) = \mathbf{1}_A(a) = \alpha(\Phi(a)), \qquad \hat{\beta}(a) = 1 - \hat{\alpha}(a).$$

For every slot,

$$\boxed{\hat{\alpha}(a)\hat{\alpha}(b) = \hat{\alpha}(a) + \hat{\alpha}(b) - 1 + \hat{\beta}(a)\hat{\beta}(b).}$$

If

$$H(m) = \left\lfloor \frac{m-1}{2} \right\rfloor$$

is the total number of unordered slots,

$$G(m) = \#\mathcal{S}_m^{AA},$$

$$X(m) = \#\mathcal{S}_m^{A\mathcal{M}},$$

and

$$C(m) = \#\mathcal{S}_m^{\mathcal{M}\mathcal{M}},$$

then

$$\boxed{H(m) = G(m) + X(m) + C(m).}$$

Double-multiplicative slots can in turn be decomposed exactly by the least-prime-factor responsibility of their two endpoints:

$$\mathcal{M} = \bigsqcup_{p} E_p, \qquad C(m) = \sum_{p,q} C_{p,q}\,(m).$$

The fixed-sum relation is thereby organized completely into a finite terminal relation geometry with no omissions.

This part ultimately establishes the standard relation-sieve toolchain

$$\boxed{\mathsf{B} \to \mathsf{P} \to \tau \to \mathsf{H} \to \mathsf{F} \to \mathsf{O} \to \mathsf{C} \to \mathsf{Sat},}$$

representing birth, projection, terminality, defect, first defect, opportunity, certificate, and saturation, respectively.

These tools elevate “sieving” from a process of deleting integer positions to a complete theory of preserving, finitely controlling, and terminally deciding relation fibers.

Keywords: finite-control geometry; multiplicative-depth tensor; atomic vertex; relation orbit; local obstruction; local admissibility; CRT survival tree; first local elimination; genuine death; relation alphabet; relation-faithful sieve; odd–even dual channel; terminal saturation; natural terminal segment; exact additive–multiplicative completion

Part I Finite-Control Geometry

Chapter 1 From Relational Language to Control Geometry

Part III wrote an arbitrary finite prime relation as

$$R_\Theta(x) = \mathcal{B}_\Theta(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_r(x;\Theta))).$$

This definition answers:

Which positions does a relation read, and how are the identities at those positions combined?

The present part fixes a finite control depth and studies how multiplicative structure acts on the entire relation.

A common misunderstanding must be avoided here.

The fact that a prime q divides one endpoint does not mean that "the entire relation is deleted by q."

A relation instance loses its all-prime identity at the q-layer only when the relation predicate requires that endpoint to be prime and that endpoint is in fact a multiple of q larger than q.

Thus the objects of control geometry are not merely "sieved integers," but

finite endpoint tuples carrying relation positions.

Chapter 2 Relation Spaces of Arbitrary Finite Order

Let the positive odd integers be

$$\mathbb{O}_{\geq 3} = \{3,5,7,\dots\}.$$

For relation order $r \geq 1$, define

$$\boxed{\mathcal{X}_r = \mathbb{O}_{\geq 3}^{r}.} \tag{2.1}$$

A point

$$x = (x_1, \dots, x_r)$$

represents an ordered r-endpoint relation instance.

If a problem uses unordered endpoints, one may further quotient by the appropriate permutation symmetry; the underlying control structure is unchanged.

2.1 Prime-Gap Cells

Let the odd primes be ordered as

$$p_1 < p_2 < p_3 < \cdots.$$

Define the one-dimensional prime-gap cell

$$\boxed{I_n = \{p_n, p_n + 2, \dots, p_{n+1}\}.} \tag{2.2}$$

For a multi-index

$$\boldsymbol{i} = (i_1, \dots, i_r),$$

define the higher-dimensional relation cell

$$\boxed{\mathcal{C}_{\boldsymbol{i}} = I_{i_1} \times \cdots \times I_{i_r}.} \tag{2.3}$$

Then

$$\boxed{\mathcal{X}_r = \bigcup_{\boldsymbol{i}} \mathcal{C}_{\boldsymbol{i}}.} \tag{2.4}$$

This gives the relation space a natural decomposition into finite cells.

Chapter 3 The Multiplicative-Depth Tensor

For an integer $n > 1$, let $\Omega(n)$ be the total number of prime factors counted with multiplicity.

Define

$$\boxed{\Delta_r(x_1, \dots, x_r) = (\Omega(x_1), \dots, \Omega(x_r)).} \tag{3.1}$$

It records the multiplicative depth of every endpoint.

When all endpoints are odd primes,

$$\Omega(x_j) = 1 \quad (1 \le j \le r).$$

Therefore define the atomic vertex skeleton

$$\boxed{\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1).} \tag{3.2}$$

Clearly,

$$\boxed{\mathcal{V}_r = \mathbb{P}^r_{\text{odd}}.} \tag{3.3}$$

Hence

every finite-order simultaneous-primality relation is an atomic-vertex relation in the same multiplicative-depth field.

(3.4)

Chapter 4 Vertices, Edges, Faces, and Shallow Multiplicative Depth

Inside a cell

$$C_i,$$

if a coordinate lies exactly at a prime endpoint, that coordinate has multiplicative depth 1.

If a coordinate lies strictly between two consecutive primes, it is an odd composite and therefore has depth greater than 1.

Thus

position in a relation cell	multiplicative-depth structure
atomic vertex	$(1, \dots, 1)$
edge interior	at least one component > 1
two-dimensional face interior	at least two components > 1
⋮	⋮

(4.1)

Prime and shallow almost-prime relations therefore require no separate relation spaces.

If the allowed depth set is

$$B \subseteq \mathbb{N}^r,$$

then the target region is simply

$$\boxed{\Delta_r^{-1}(B).} \tag{4.2}$$

The all-prime relation corresponds to

$$B = \{(1, \dots, 1)\}.$$

Chapter 5 Relation Orbits

Let the parameter space be

$$\mathcal{U}.$$

Given a map

$$\boxed{F = (F_1, \dots, F_r): \mathcal{U} \to \mathcal{X}_r,} \tag{5.1}$$

define its image

$$\boxed{\Gamma_F = F(\mathcal{U}).} \tag{5.2}$$

and call it a **relation orbit**.

Depending on the form of F, a relation orbit may be

- a fixed-sum reflection line;
- a fixed-difference translation line;
- an affine subspace;
- a translation orbit of a finite pattern;
- a polynomial curve;
- a sparse exponential orbit;
- a family of arithmetic chains;
- a family of intervals or windows;
- a complete factor-identity block.

What unifies them is that all enter the same fixed identity spectrum and the same multiplicative-depth field.

Chapter 6 Atomic-Vertex Preimage Theorem

Theorem 6.1

For a relation map

$$F = (F_1, \dots, F_r),$$

we have

$$\boxed{\{t \in \mathcal{U}: F_1(t), \dots, F_r(t) \text{ are all odd primes}\} = F^{-1}(\mathcal{V}_r).} \tag{6.1}$$

Proof

Since

$$\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1),$$

we have

$$F(t) \in \mathcal{V}_r$$

if and only if

$$\Omega(F_j(t)) = 1$$

for every j.

This is exactly the statement that all $F_j(t)$ are odd primes. This proves the theorem.

More generally,

$$\boxed{\{t: \Delta_r(F(t)) \in B\} = F^{-1}(\Delta_r^{-1}(B)).} \tag{6.2}$$

This is the exact interface between relation orbits and the multiplicative-depth field.

Chapter 7 Finite Control Depth

Fix a real number

$$z \geq 2.$$

Define the set of control primes

$$\boxed{\mathcal{P}(z) = \{q \in \mathbb{P} : q \leq z\}.} \tag{7.1}$$

and the primorial

$$\boxed{P(z) = \prod_{q \leq z} q\,.} \tag{7.2}$$

The role of a finite control depth z is:

> to use only the prime multiplicative layers up to z to perform finite identity readouts on the relation endpoints.

It is not synonymous with terminal depth.

If endpoints may be very large, z may be only a presieving layer; if

$$z \geq \sqrt{X},$$

then for all endpoints not exceeding X, it becomes a complete primality-terminal layer.

Chapter 8 Phase Periodicity and the Finite Control Period

For a fixed finite control layer $\mathcal{P}(z)$, each divisibility condition

$$q \mid n$$

depends only on

$$n \bmod q.$$

Hence all local phases repeat jointly modulo

$$\boxed{W_z = \prod_{\substack{3\le q\le z \\ q\in\mathbb{P}}} q\,.} \tag{8.1}$$

When read in odd-number atomic coordinates, the corresponding complete period on the ordinary integer axis is $2W_z$.

Two kinds of "seam" must be distinguished here:

phase seam: the finite control layer returns to the same residue phase;

generation seam: a new control prime begins to have an exclusive least-factor role at its own square scale.

They answer, respectively:

> *When do we return to the same phase?*

and

> *When is a genuinely new control layer added?*

Figure 4-01 Overview of Finite-Control Geometry

The controlled object is not an isolated integer but a finite tuple with relational positions; the multiplicative-depth field places atomic vertices, shallow almost-primes, and general relation orbits in one finite geometry.

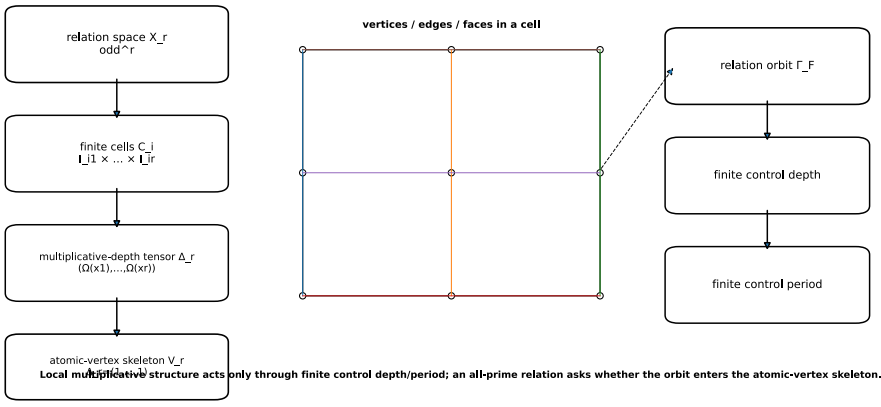


Figure 4-01. Overview of Finite-Control Geometry. Finite-control geometry places an r-endpoint object with relational positions in $\mathcal{X}_r = \mathbb{O}^r_{\ge 3}$, decomposes the space into prime-gap cells, labels endpoint depths with the multiplicative-depth tensor Δ_r, and identifies all-prime relations with the atomic-vertex skeleton $\mathcal{V}_r = \Delta_r^{-1}(1,\dots,1)$. Relation orbits, finite control depth, and finite control period then operate in this same geometry.

Part II Local Obstruction and Admissibility

Chapter 9 Unified Definition of Local Obstruction

For a relation orbit

$$F = (F_1, \dots, F_r)$$

and a prime ℓ, define

$$\boxed{B_\ell(F) = \left\{ t \bmod \ell : \prod_{j=1}^{r} F_j(t) \equiv 0 \pmod{\ell} \right\}.} \tag{9.1}$$

This means:

among the parameter positions modulo ℓ, at least one relation endpoint lies in the divisibility class of ℓ.

If

$$B_\ell(F)$$

covers every parameter residue class, then regardless of the parameter choice at least one endpoint is divisible by ℓ.

Therefore

local obstruction means that the family of divisibility rays of one multiplicative atom completely covers the relation parameter space.

(9.2)

Chapter 10 Local Admissibility Does Not Equal Global Occupancy

If no prime ℓ produces complete coverage, the relation orbit is said to satisfy **local admissibility**. This means only

> no single finite-modulus multiplicative structure permanently seals off the entire relation orbit.

(10.1)

It does not automatically imply

> the relation orbit actually enters the atomic vertex skeleton.

(10.2)

Thus one must distinguish rigorously between

$$\boxed{\text{local admissibility} \quad \text{and} \quad \text{global occupancy.}} \tag{10.3}$$

The former is a local necessary condition;

the latter is a global relation fact.

Chapter 11 Local Fingerprint of a Finite Offset Pattern

Let

$$\boxed{H = \{h_1, \dots, h_k\}.} \tag{11.1}$$

be a finite offset pattern, corresponding to the relation

$$n \mapsto (n + h_1, \dots, n + h_k).$$

For a prime q, define the number of distinct residues occupied by the pattern modulo q:

$$\boxed{\nu_H(q) = \#\{h \bmod q : h \in H\}.} \tag{11.2}$$

Define the number of admissible residues by

$$\boxed{\nu_H(q) = q - \nu_H(q).} \tag{11.3}$$

If

$$\nu_H(q) = q,$$

then H covers every residue class modulo q.

For every n, at least one

$$n + h$$

is therefore divisible by q.

If that endpoint is larger than q, it must be composite.

Hence

$$\boxed{\nu_H(q) = q \Rightarrow \text{the all-prime pattern can occur only through finitely many boundary exceptions.}} \tag{11.4}$$

A boundary exception occurs when an endpoint divisible by q is exactly q itself.

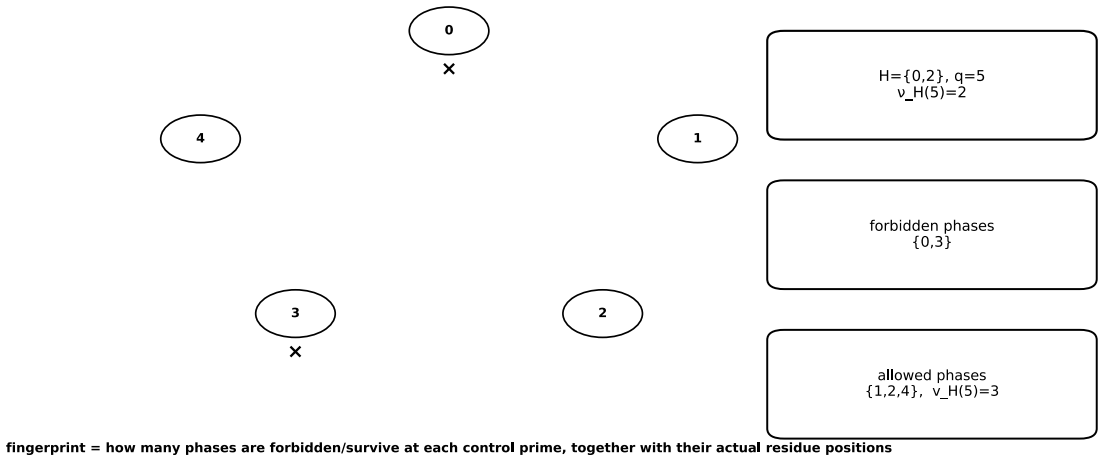


Figure 4-02. Local Obstruction Fingerprint. For a finite offset pattern H and a prime q, the number of occupied residue classes is $\nu_H(q)$, while the number of allowed classes is $\nu_H(q) = q - \nu_H(q)$. For $H = \{0,2\}$ modulo 5, the forbidden starting phases are $\{0,3\}$ and the surviving phases are $\{1,2,4\}$. The local fingerprint preserves not only the counts but the actual residue positions.

Chapter 12 Local Admissibility of a Pattern

Definition 12.1

If for every prime q,

$$\boxed{\nu_H(q) < q,} \tag{12.1}$$

then the pattern H is called **locally admissible**.

This has the same mathematical content as classical admissibility, but here it is interpreted directly as

> no single least multiplicative layer can cover every translation position.

(12.2)

Theorem 12.2 Finite-Check Theorem

If

$$q > |H|,$$

then

$$\nu_H(q) \leq |H| < q.$$

Therefore, to determine whether a finite pattern admits complete local coverage, it is enough to check

$$\boxed{q \leq |H|.} \tag{12.3}$$

Proof

The pattern has at most $|H|$ distinct offsets and therefore occupies at most $|H|$ distinct residue classes modulo q.

If $q > |H|$, it cannot cover all q residue classes. This proves the theorem.

Chapter 13 A Completely Obstructed Pattern

Consider

$$H = \{0,2,4\}.$$

Modulo 3,

$$0,\ 2,\ 1$$

cover all three residue classes.

Hence

$$\nu_H(3) = 3.$$

For every integer n, at least one of

$$n,\ n+2,\ n+4$$

is divisible by 3.

Apart from finitely many boundary cases, the three numbers therefore cannot all be prime simultaneously.

Thus

$$\boxed{H = \{0,2,4\}}$$

is one of the simplest examples of complete local obstruction.

Chapter 14 Local Capacity of Arithmetic Patterns

Consider

$$H_{k,d} = \{0, d, 2d, \dots, (k-1)d\}.$$

Fix a prime $q \le k$.

If

$$q \nmid d,$$

then d is invertible modulo q, and the first q terms already run through every residue class modulo q.

Therefore

$$\nu_{H_{k,d}}(q) = q.$$

Hence, if a fixed arithmetic pattern is to satisfy local admissibility for every

$$q \le k,$$

it is necessary that

$$\boxed{q \mid d \qquad \text{for every prime } q \le k.} \tag{14.1}$$

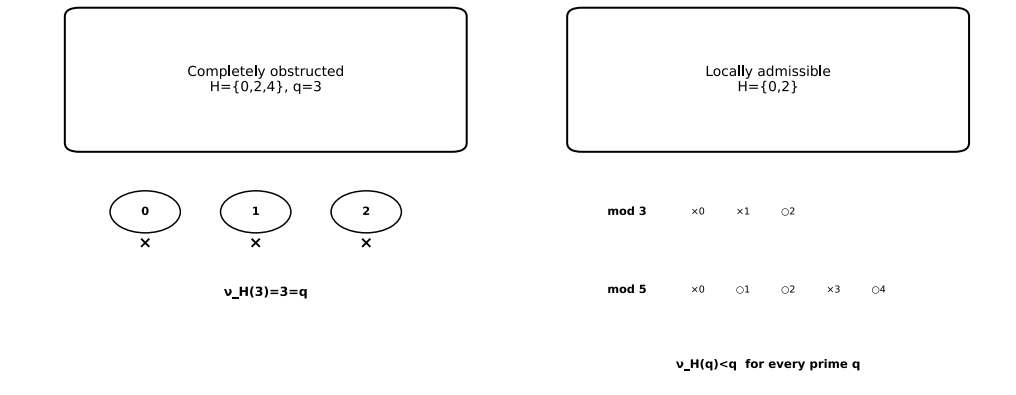
Thus the minimal natural common step contains

$$\boxed{W_k = \prod_{q \le k} q\,.} \tag{14.2}$$

This connects directly with the necessary step condition for Green–Tao arithmetic chains in Part III.

Figure 4-03 Completely Obstructed vs Locally Admissible Templates

Local admissibility is a deterministic finite-modulus condition, not a probability statement: either some prime layer covers every phase or every prime layer leaves at least one phase.



Local admissibility rules out only the possibility that one finite modulus permanently seals the entire template; it is not the same as global atomic occupancy.

Figure 4-03. Completely Obstructed vs Locally Admissible Templates. The pattern $H = \{0,2,4\}$ covers all three residues modulo 3, so one finite modulus completely obstructs it. The pattern

$H = \{0,2\}$, by contrast, retains at least one allowed phase modulo 3, modulo 5, and the other relevant prime layers. Local admissibility is a deterministic finite-modulus condition: it rules out a single small-prime layer permanently sealing the entire pattern, but it is not the same as global atomic occupancy.

Chapter 15 Local Capacity After Finitely Many Rounds

Let S be a finite set of control primes. Suppose the pattern has

$$v_H(q) > 0$$

for every $q \in S$.

Then the Chinese remainder theorem gives the number of admissible starting residue classes modulo

$$M_S = \prod_{q \in S} q$$

as

$$\boxed{\rho_H(S) = \prod_{q \in S} v_H(q).} \tag{15.1}$$

This number describes exactly

how many phase classes survive all the local divisibility conditions in the specified finite multiplicative layers.

But it is still not the final number of prime relations.

Some locally admissible positions may have composite endpoints built from larger prime factors.

Part III The CRT Survival Tree and First Local Elimination

Chapter 16 From Finite Capacity to the CRT Survival Tree

Chapter 15 gave

$$\rho_H(S) = \prod_{q \in S} v_H(q).$$

This is the number of jointly admissible phases after finitely many control layers.

But a product tells us only how many phases remain at the end. It does not tell us

how each layer branches, which parameters survive all the way through, or at which layer a parameter loses local admissibility for the first time.

The Chinese remainder theorem writes all these questions as an exact finite tree.

16.1 Layerwise Control Moduli

Take distinct control primes

$$\boxed{p_1 < p_2 < \cdots < p_s.} \tag{16.1}$$

Define

$$\boxed{W_i = \prod_{j=1}^{i} p_j, \qquad W_0 = 1.} \tag{16.2}$$

For a pattern

$$H = \{h_1, \ldots, h_k\},$$

define the jointly surviving residue set at layer i by

$$\boxed{A_i(H) = \{a \bmod W_i : a + h \not\equiv 0 \pmod{p_j} \text{ for all } h \in H,\ 1 \le j \le i\}.} \tag{16.3}$$

By convention,

$$A_0(H) = \{0\}.$$

16.2 Exact Branching of a Surviving Parent Node

Fix

$$a \bmod W_{i-1} \in A_{i-1}(H).$$

Modulo

$$W_i = W_{i-1} p_i,$$

it has exactly p_i child residue classes.

Because W_{i-1} and p_i are coprime, these p_i child classes run through every residue class modulo p_i.

The pattern H occupies

$$v_H(p_i)$$

forbidden residue classes modulo p_i, so exactly

$$\boxed{v_H(p_i) = p_i - \nu_H(p_i)} \tag{16.4}$$

children survive to the next layer.

Thus

$$\boxed{\text{each surviving parent node} \to p_i \text{ child nodes,}}$$

of which

$$\boxed{v_H(p_i) \text{ survive,} \qquad \nu_H(p_i) \text{ lose local admissibility for the first time at this layer.}} \tag{16.5}$$

Theorem 16.1 CRT Survival-Tree Counting Formula

For every $i \geq 1$,

$$\boxed{|A_i(H)| = \prod_{j=1}^{i} \left(p_j - \nu_H(p_j)\right) = \prod_{j=1}^{i} v_H\,(p_j).} \tag{16.6}$$

Proof

At layer 0 there is a single root node.

Every surviving node at layer $j-1$ produces exactly

$$v_H(p_j)$$

surviving children at layer j.

Multiplying these branching numbers layer by layer gives the formula. This proves the theorem.

16.3 Survival Proportion

The total number of phases is

$$W_i = \prod_{j=1}^{i} p_j.$$

Therefore

$$\boxed{\frac{|A_i(H)|}{W_i} = \prod_{j=1}^{i} \left(1 - \frac{\nu_H(p_j)}{p_j}\right).} \tag{16.7}$$

Define

$$\boxed{\eta_j = 1 - \frac{\nu_H(p_j)}{p_j}.} \tag{16.8}$$

Then

$$\boxed{\frac{|A_i(H)|}{W_i} = \prod_{j=1}^{i} \eta_j\,.} \tag{16.9}$$

The factor η_j is the exact proportion of children that survive the j-th finite control layer for each previously surviving parent node.

16.4 Complete Obstruction Means Termination of the Entire Tree

If some p_i satisfies

$$v_H(p_i) = p_i,$$

then

$$v_H(p_i) = 0.$$

Hence

$$\boxed{A_i(H) = \varnothing.}$$

All subsequent layers are empty as well.

Therefore

complete local obstruction is exactly the termination of the whole CRT survival tree at a finite layer.

(16.10)

Conversely, as long as every incorporated layer satisfies

$$v_H(p_i) > 0,$$

the finite tree retains surviving branches.

Those branches represent finite local phases that remain admissible; the terminal identities of the actual integer endpoints must still be resolved through the genuine-death layers.

Figure 4-04 CRT Survival Tree

Example template H={0,2}: impose local admissibility modulo 3, then modulo 5; CRT lifts the surviving phases exactly to modulus 15.

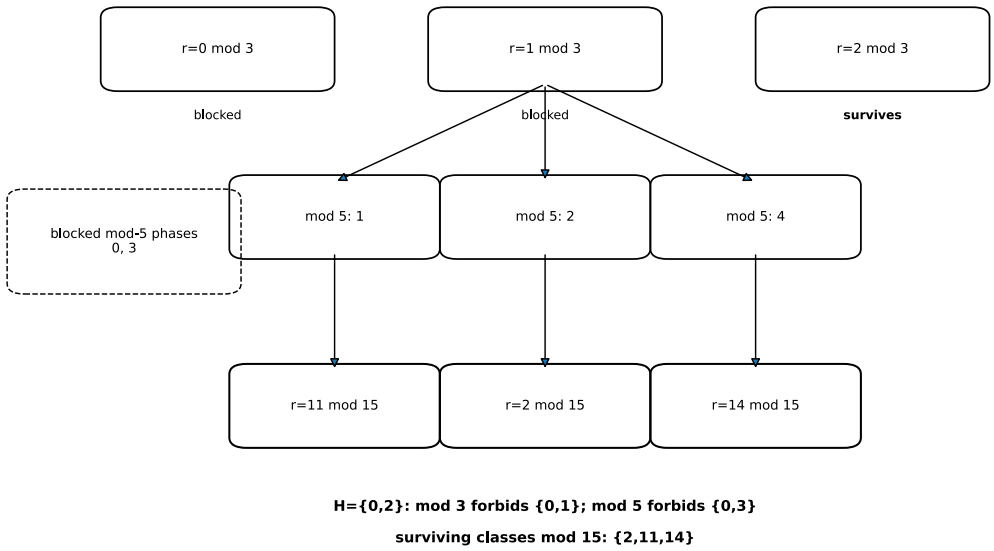


Figure 4-04. CRT Survival Tree. For the example template $H = \{0,2\}$, local admissibility is first imposed modulo 3, then modulo 5. The Chinese remainder theorem lifts the surviving phases

without loss to the surviving classes $\{2,11,14\}$ modulo 15, producing an exact finite CRT survival tree.

Chapter 17 Conservation of First Local Elimination

The CRT survival tree not only counts the branches that survive to the final layer. It also partitions every parameter phase modulo W_s according to the layer at which failure first occurs.

17.1 First-Local-Elimination Sets

Fix the final control layer s.

Define

$$\boxed{F_i(H) = \left\{a \bmod W_s : \begin{matrix} a \text{ survives all layers } p_1, \dots, p_{i-1}, \\ a \text{ fails for the first time at layer } p_i \end{matrix}\right\}.} \tag{17.1}$$

After lifting the final joint-survivor set to modulus W_s, continue to denote it by

$$A_s(H).$$

Then

$$\boxed{\mathbb{Z}/W_s\mathbb{Z} = A_s(H) \sqcup F_1(H) \sqcup \cdots \sqcup F_s(H).} \tag{17.2}$$

Every parameter phase either passes through all incorporated control layers or has a unique layer of first local elimination.

17.2 Exact Count of First Eliminations

The proportion surviving the first $i-1$ layers is

$$\prod_{j<i} \eta_j.$$

The proportion failing at layer i is

$$1 - \eta_i = \frac{\nu_H(p_i)}{p_i}.$$

Therefore

$$\boxed{|F_i(H)| = W_s(1-\eta_i)\prod_{j<i} \eta_j\,.} \tag{17.3}$$

The number of final survivors is

$$\boxed{|A_s(H)| = W_s \prod_{i=1}^{s} \eta_i\,.} \tag{17.4}$$

Theorem 17.1 Conservation of First Local Elimination

We have

$$\boxed{1 = \prod_{i=1}^{s} \eta_i + \sum_{i=1}^{s} (1-\eta_i)\prod_{j<i} \eta_j\,.} \tag{17.5}$$

Equivalently,

$$\boxed{W_s = |A_s(H)| + \sum_{i=1}^{s} |F_i(H)|.} \tag{17.6}$$

Proof

The counting identity follows immediately from the disjoint partition

$$\mathbb{Z}/W_s\mathbb{Z} = A_s(H) \sqcup \bigsqcup_{i=1}^{s} F_i(H).$$

Dividing by W_s, and substituting (17.3) and (17.4), gives (17.5). This proves the theorem.

17.3 From a Phase First Killer to an Integer First Killer

Equation (17.5) can be read as

$$\boxed{\text{all parameter phases} = \text{joint survival} + \text{first local elimination at each layer.}} \tag{17.7}$$

This preserves one crucial item of information beyond the final survival proportion:

$$\boxed{\text{where failure first occurred.}}$$

Thus the CRT survival tree already has a phase-level first killer.

But a phase-level first killer and genuine integer death belong to different layers:

- first local elimination records the first small-prime divisibility triggered by a parameter residue class;
- genuine death records the unique least prime factor of a concrete integer;
- the relation first killer then aggregates the genuine deaths of several endpoints into the first responsible prime at relation level.

The finite-control chain therefore becomes

$$\boxed{\text{local admissibility} \rightarrow \text{CRT survival tree} \rightarrow \text{first local elimination} \rightarrow \text{genuine death} \rightarrow \text{relation fi}} \tag{17.8}$$

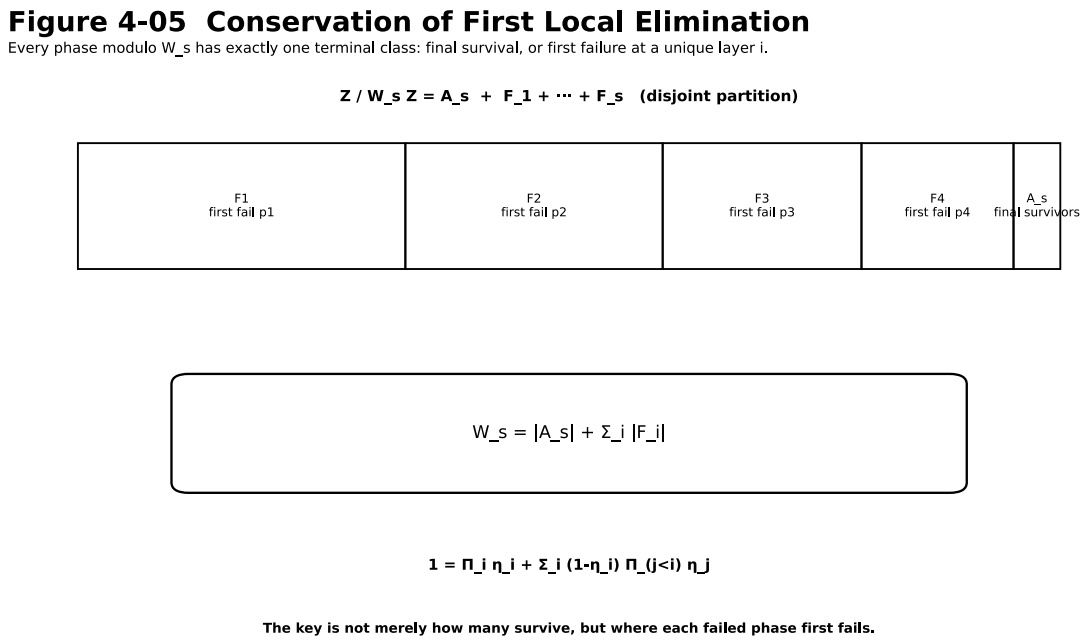


Figure 4-05. Conservation of First Local Elimination. After fixing the final control layer s, all parameter phases modulo W_s partition exactly into the final joint-survival set $A_s(H)$ and the first-elimination sets $F_i(H)$. Hence $W_s = |A_s(H)| + \sum_i |F_i(H)|$. This ledger is stronger than retaining only the final survival proportion because it records the first layer at which every failed phase fails.

Part IV Genuine Death, Relation Alphabets, and Relation-Faithful Sieves

Chapter 18 Ordinary Divisibility and Genuine Death

If an odd composite

$$n$$

is divisible by several primes, an ordinary sieve will hit it at several layers.

To obtain a unique relation history, one must specify the prime layer that first certifies it as composite.

Define

$$\boxed{\kappa(n) = P^{-}(n).} \tag{18.1}$$

Definition 18.1 Genuine-Death Layer

For an odd prime q, define

$$\boxed{D_q = \{n \text{ odd composite}: P^{-}(n) = q\}.} \tag{18.2}$$

Then

$$\boxed{D_q \cap D_r = \varnothing \qquad (q \neq r).} \tag{18.3}$$

Every odd composite belongs to exactly one D_q.

Therefore

every odd composite genuinely dies exactly once, at the layer of its least prime factor.

(18.4)

Chapter 19 The Relation First Killer

For a finite all-prime-type relation instance

$$R = (n_1, \dots, n_r),$$

if at least one endpoint is composite, define

$$\boxed{\kappa(R) = \min\{P^-(n_j) : n_j \text{ composite}\}.} \tag{19.1}$$

This is the **first responsible prime** of the relation.

If all endpoints are prime, set

$$\kappa(R) = \infty.$$

Thus a candidate relation has only two terminal outcomes:

$$\boxed{\begin{cases} \kappa(R) < \infty, & \text{it genuinely dies at its unique first-killer layer;} \\ \kappa(R) = \infty, & \text{all endpoints are atoms.} \end{cases}} \tag{19.2}$$

This gives the relation sieve a non-overlapping death ledger.

Figure 4-06 Genuine Death and the Relation First-Killer

Element-level genuine death is uniquely assigned to the least prime factor; the relation first-killer is the minimum responsibility prime among all composite endpoints.

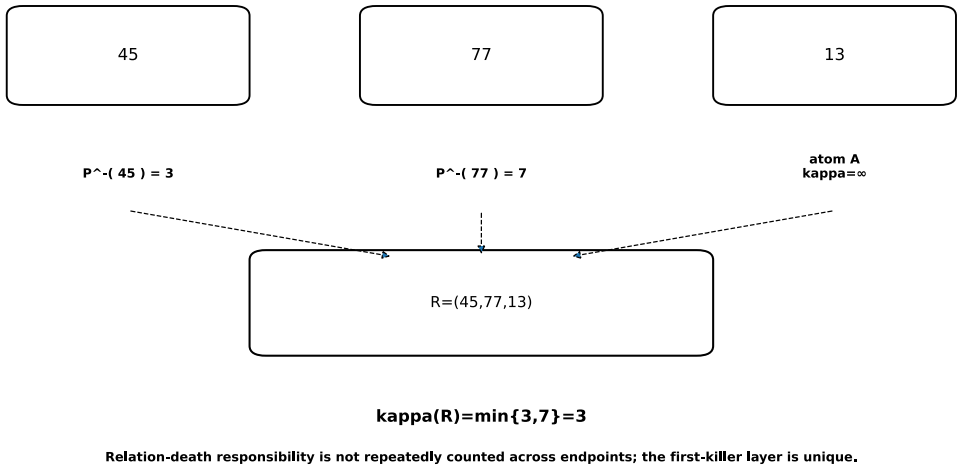


Figure 4-06. Genuine Death and the Relation First-Killer. Element-level genuine death is uniquely assigned to the least prime factor $P^-(n)$. For a candidate relation $R = (n_1, \dots, n_r)$, the relation first-killer is the minimum among the least prime factors of all composite endpoints; if all endpoints are prime, then $\kappa(R) = \infty$. Relation death therefore also has a unique first responsibility layer.

Chapter 20 Relation Alphabets

The relation first killer records where a candidate relation first fails.

After terminality, one may ask a further question:

> *Into which combinations of identities can a given class of relations ultimately fall?*

This gives a finite relation alphabet.

20.1 Endpoint-Identity Letters

Define the coarse terminal identity map

$$\boxed{\iota(n) = \begin{cases} A, & n \text{ is an atom;} \\ \mathcal{M}, & n \text{ is multiplicatively occupied;} \\ \partial, & n \text{ lies outside the legal positive-integer domain of the current relation.} \end{cases}} \tag{20.1}$$

If the least-prime-factor responsibility must be retained, define instead

$$\boxed{\iota^{\sharp}(n) = \begin{cases} A, & n \in A; \\ \mathcal{M}_q, & P^{-}(n) = q; \\ \partial, & n \text{ lies outside the legal domain.} \end{cases}} \tag{20.2}$$

20.2 Relation Identity Words

For an r-endpoint relation instance

$$R = (n_1, \dots, n_r),$$

define

$$\boxed{\omega(R) = (\iota(n_1), \dots, \iota(n_r)).} \tag{20.3}$$

Its refined identity word is

$$\boxed{\omega^{\sharp}(R) = \left(\iota^{\sharp}(n_1), \dots, \iota^{\sharp}(n_r)\right).} \tag{20.4}$$

For a relation family $\mathcal{R}$, define the terminal relation alphabet

$$\boxed{\Sigma_{\mathcal{R}} = \{\omega(R) : R \in \mathcal{R}\}.} \tag{20.5}$$

and the refined alphabet

$$\boxed{\Sigma_{\mathcal{R}}^{\sharp} = \{\omega^{\sharp}(R) : R \in \mathcal{R}\}.} \tag{20.6}$$

Thus

the relation alphabet is the finite language of all terminal identity combinations available to a class of relations. (20.7)

20.3 Coarse Alphabet for Fixed Sums

Fixed sums use unordered binary slots.

Within the legal positive odd endpoints, only three coarse terminal types occur:

$$\boxed{AA, \qquad A\mathcal{M}, \qquad \mathcal{M}\mathcal{M}.} \tag{20.8}$$

Here

$$A\mathcal{M}$$

contains both ordered forms

$$A\mathcal{M} \quad \text{and} \quad \mathcal{M}A.$$

Therefore

$$\boxed{\Sigma_{\text{sum}} = \{AA, A\mathcal{M}, \mathcal{M}\mathcal{M}\}.} \tag{20.9}$$

The exact additive–multiplicative completion in Part VII is precisely a complete count of these three terminal words.

20.4 Purified Alphabet for Twin in a Single Square Horizon

For the unit-difference relation

$$(n, n+2),$$

suppose it has escaped all previous control primes, and let the current final control prime be $r > 2$. Then an old double-escape relation can have only the following terminal types within the current square horizon:

$$\boxed{AA, \qquad \mathcal{M}_r A, \qquad A\mathcal{M}_r.} \tag{20.10}$$

The type

$$\mathcal{M}_r \mathcal{M}_r$$

cannot occur, because it would require

$$r \mid n, \qquad r \mid n+2,$$

and hence

$$r \mid 2,$$

contrary to $r > 2$.

Thus the relation alphabet at this purification layer is

$$\boxed{\Sigma_{\text{Twin}}^{(r)} = \{AA, \mathcal{M}_r A, A\mathcal{M}_r\}.} \tag{20.11}$$

The qualification is essential: this alphabet describes "old double escape + current single-layer purification," not the global coarse alphabet of all integer pairs differing by 2.

20.5 Relation Alphabets and Relation Fidelity

A relation alphabet connects

$$\boxed{\text{endpoint identities} \rightarrow \text{terminal relation types} \rightarrow \text{executable states.}}$$

A relation-faithful compression must therefore satisfy at least

distinct relation identity words on which the target proposition depends may not be irreversibly merged.

(20.12)

This directly supports

- the genuine-death ledger;
- fixed-sum slot classification;

- general finite patterns;
- complete factor-identity relations;
- finite-state output in the RSL.

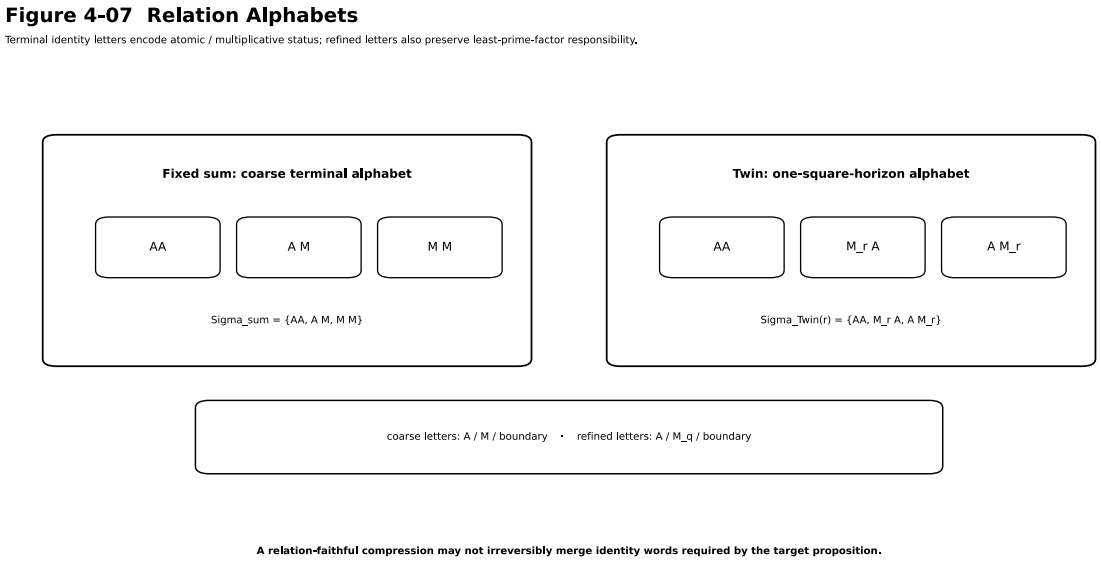


Figure 4-07. Relation Alphabets. The coarse endpoint letters are $A, \mathcal{M}, \partial$; the refined letters additionally retain least-prime-factor responsibility as $\mathcal{M}_q$. The coarse terminal alphabet for fixed sums is $\{AA, A\mathcal{M}, \mathcal{M}\mathcal{M}\}$. Under the 'old double escape + current single-layer purification' condition, the one-square-horizon Twin alphabet is $\{AA, \mathcal{M}_r A, A\mathcal{M}_r\}$. A relation-faithful compression may not irreversibly merge identity words that the target proposition needs to distinguish.

Chapter 21 The Relation-Faithful Sieve

Definition 21.1

A relation-sieve state records at least

$$\text{relation-slot positions} + \text{endpoints} + \text{current endpoint identities} + \text{first killer} + \text{necessary birth a} \tag{21.1}$$

If this information is sufficient to determine the exact update at the next control layer and to distinguish the relation identity words on which the target proposition depends, the state is called **relation-faithful**.

Relation fidelity does not require preserving all raw data forever.

It requires only that

no compression may delete relation information genuinely required by the target proposition.

(21.2)

Chapter 22 Why “How Many Remain?” Is Not a Sufficient State

Fix an even integer N, and define the unordered relation slots

$$\mathcal{S}_N = \{(x, N-x) : 3 \le x \le N/2,\ x \text{ odd}\}.$$

Consider

$$N_1 = 2006, \qquad N_2 = 2008.$$

Both have

$$501$$

initial slots.

After processing

$$q = 3,$$

both leave

$$168$$

relation slots.

If the state stores only

$$(H, |S_3|),$$

the two targets are identical.

But at the next layer,

$$q = 5,$$

N_1 has 66 newly and genuinely deleted slots,

whereas N_2 has 67.

The next states are therefore, respectively,

$$102, \qquad 101.$$

Hence

$$\boxed{(H, |S_q|) \text{ does not determine the next-layer update.}} \tag{22.1}$$

This shows that

the concrete positions and residue configurations of relation slots carry information that scalar counts cannot preserve.

(22.2)

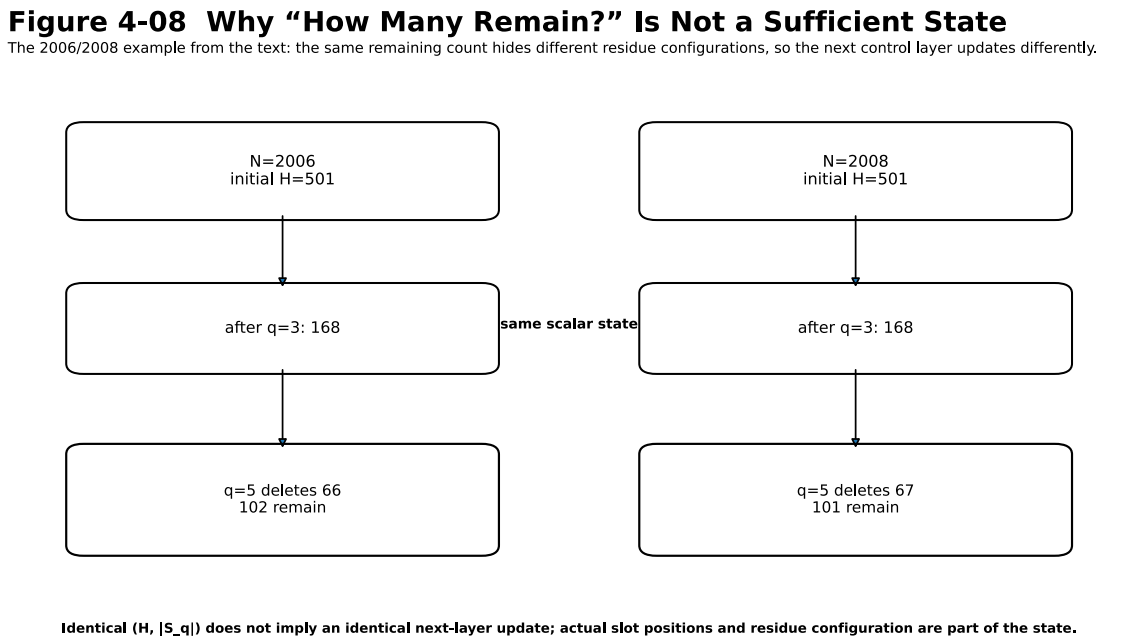


Figure 4-08. Why “How Many Remain?” Is Not a Sufficient State. In the fixed-sum relation, $N_1 = 2006$ and $N_2 = 2008$ both begin with 501 slots and both leave 168 slots after $q = 3$. At $q = 5$, however, they newly and genuinely delete 66 and 67 slots respectively, leaving 102 and 101. Thus the same $(H, |S_q|)$ is insufficient to determine the next-layer update; actual slot positions and residue configurations must be part of a relation-faithful state.

Chapter 23 Fixed Sum as a Two-Point Reflection Correlation

For any $0/1$-valued function f, define the fixed-sum relation quantity

$$\boxed{R_f(N) = \sum_x f(x) f(N-x).} \tag{23.1}$$

Define the reflection operator

$$\boxed{J_N f(x) = f(N-x).} \tag{23.2}$$

Then

$$\boxed{R_f(N) = \langle f, J_N f \rangle.} \tag{23.3}$$

Thus a fixed sum is not a one-point statistic but a two-point reflection correlation.

This is why total counts, densities, or finite-modulus marginal distributions generally do not determine a fixed-sum relation.

Chapter 24 Finite-Modulus Data Do Not Determine Reflection Pairing

Take

$$W = 2 \cdot 3 \cdot 5 \cdot 7 = 210,$$

and fix

$$N = 700.$$

Consider the sets

$$A = \{11,689\},$$

$$B = \{221,689\}.$$

Because

$$221 = 11 + 210,$$

11 and 221 have the same residues modulo

$$2,3,5,7.$$

Therefore A and B have the same finite-modulus residue multisets.
But

$$11 + 689 = 700,$$

whereas

$$221 + 689 \neq 700.$$

Counting ordered relations,

$$R_A(700) = 2, \qquad R_B(700) = 0.$$

Thus

identical finite-modulus local information does not determine a specified fixed-sum relation.

(24.1)

Any relation tool that discards the concrete phase correspondence

$$x \leftrightarrow N - x$$

must recover that information elsewhere.

Chapter 25 Division of Labor Between Relation Sieves and Element Sieves

An element sieve is well suited to answering

> which integer positions have already been occupied by multiplicative factors?

A relation sieve asks further

> how do these endpoints pair, die, and become terminal within the specified relation?

This gives a two-layer structure:

$$\boxed{\begin{array}{c}\text{element layer}\\ \text{endpoint identity}\\ \Downarrow\\ \text{relation layer}\\ \text{relation-fiber identity}\end{array}} \tag{25.1}$$

A relation sieve is not a second sieving power.

What changes is the object being sieved:

$$\boxed{\text{from integer positions to relation fibers.}} \tag{25.2}$$

Part V Parity Channels of Multiplicative Depth

Chapter 26 Liouville Parity Identity

Define

$$\boxed{\lambda(n) = (-1)^{\Omega(n)}.} \tag{26.1}$$

Then

$$\lambda(n) = -1$$

means odd multiplicative depth, while

$$\lambda(n) = 1$$

means even multiplicative depth.

Primes lie at odd depth 1.

Chapter 27 Parity Flip Under Genuine Death

Suppose

$$n = qm,$$

with

$$q = P^{-}(n).$$

Then

$$\Omega(n) = 1 + \Omega(m).$$

Therefore

$$\boxed{\lambda(n) = -\lambda(m).} \tag{27.1}$$

A genuine-death layer thus does more than remove n; it records the multiplicative-depth flip from the cofactor m to qm.

Chapter 28 Odd–Even Dual-Channel Transfer

Suppose that immediately before control depth q, genuine survivors are separated into

$$O_{q^-}$$

and

$$E_{q^-},$$

denoting the odd- and even-multiplicative-depth channels.

Let

$$T_q$$

denote the genuine-death transfer associated with "multiplication by the new least prime factor q."

Because multiplicative-depth parity flips, we have

$$\boxed{O_q = O_{q^-} - T_q E_{q^-},} \tag{28.1}$$

$$\boxed{E_q = E_{q^-} - T_q O_{q^-}.} \tag{28.2}$$

This structure shows that

genuine death is not anonymous deletion;

it carries an exact transfer of multiplicative depth.

(28.3)

Chapter 29 Four Relation Channels

For a binary relation one may further distinguish

$$O * O, \qquad O * E, \qquad E * O, \qquad E * E.$$

A fixed-sum relation can therefore record not merely “survived/died” but the parity combination of the multiplicative depths of its two endpoints.

This can substantially strengthen finite identity certification.

It still does not replace the concrete reflection relation.

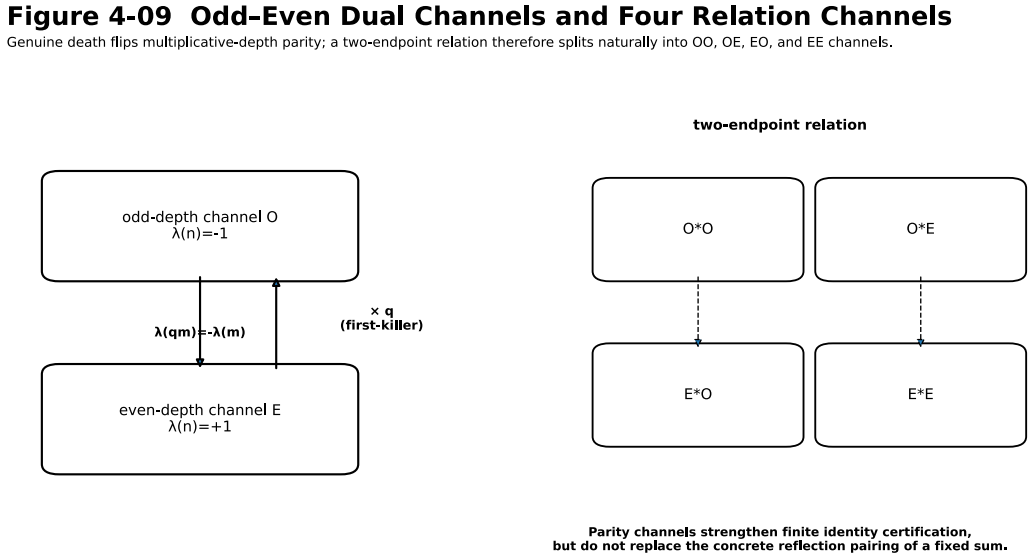


Figure 4-09. Odd–Even Dual Channels and Four Relation Channels. The Liouville identity $\lambda(n) = (-1)^{\Omega(n)}$ splits genuine survivors into odd- and even-multiplicative-depth channels. When a new least prime factor q causes genuine death, multiplicative-depth parity flips. A two-endpoint relation therefore further splits into the four channels $O * O, O * E, E * O, E * E$. This structure strengthens finite identity certification, but it does not replace the concrete reflection pairing in a fixed-sum relation.

Chapter 30 Upper Bound on Multiplicative Depth

If

$$n \leq X,$$

and every prime factor of n is larger than z, then

$$n > z^{\Omega(n)}.$$

Therefore

$$\boxed{\Omega(n) < \frac{\log X}{\log z}} \tag{30.1}$$

with the bound interpreted at integer depth.

This gives a direct relation between control depth and the remaining multiplicative complexity.

Chapter 31 Cube-Root Certification of the Odd Channel

Theorem 31.1

Suppose

$$z \geq X^{1/3},$$

and

$$3 \leq n \leq X$$

survives every genuine-death layer up to z, with

$$\lambda(n) = -1.$$

Then n must be prime.

Proof

If n were composite and had odd multiplicative depth, then

$$\Omega(n) \geq 3.$$

Since all prime factors of n exceed z,

$$n > z^3 \geq X,$$

contradicting $n \leq X$.

Thus necessarily

$$\Omega(n) = 1,$$

so n is prime. This proves the theorem.

Hence

$$\boxed{O_z(n) = \mathbf{1}_{\mathbb{P}}(n), \qquad 3 \leq n \leq X, \quad z \geq X^{1/3}.} \tag{31.1}$$

Chapter 32 The Hierarchical Difference Between Cube-Root Certification and Fixed-Sum Existence

If

$$z \geq N^{1/3},$$

then throughout $[3, N]$ the odd channel already coincides with the prime indicator.

Therefore

$$(O_z * O_z)(N) = (\mathbf{1}_{\mathbb{P}} * \mathbf{1}_{\mathbb{P}})(N).$$

Nonemptiness of the fixed-sum fiber is read directly from

$$\boxed{(O_z * O_z)(N) > 0.} \tag{32.1}$$

Cube-root certification answers

which positions are already certified prime?

whereas fixed-sum existence asks

do the correct two primes lie in the same reflection fiber?

Therefore

shallower prime certification does not automatically produce a positive lower bound for a relation.

(32.2)

Chapter 33 The P_1/P_3 Decomposition at the $X^{1/4}$-Layer

Suppose

$$z \geq X^{1/4}.$$

If a genuinely surviving composite in the odd multiplicative-depth channel exists, its depth is at least 3, but it cannot reach 5, because

$$n > z^5 \geq X^{5/4} > X.$$

Thus the odd channel can contain only

$$P_1 \quad \text{or} \quad P_3,$$

namely

$$\boxed{O_z = P_1 \sqcup P_3.} \tag{33.1}$$

For fixed sums,

$$\boxed{(O_z * O_z)(N) = (P_1 * P_1)(N) + 2(P_1 * P_3)(N) + (P_3 * P_3)(N).} \tag{33.2}$$

Thus

$$(O_z * O_z)(N) > 0$$

does not directly imply

$$(P_1 * P_1)(N) > 0.$$

The preterminal relation mass may still contain depth-3 contamination.

Chapter 34 Why a Simple Monotone Proportion Is Not Enough

In a relation state that distinguishes odd and even multiplicative depths, it is natural to search for a simple normalized quantity that might vary monotonically across sieve layers.

Examples include

- Liouville correlation;
- the product of endpoint Liouville means;
- a determinant-like quantity of the 2×2 parity relation matrix;
- the proportion of the OO channel.

None of these quantities can, in general, carry the complete relation state.

The reason is not that they contain no information, but that

the stable structure lies in relation transfer and fixed identities themselves, not in a single scalar ratio.

(34.1)

The principal state of a relation sieve must therefore continue to preserve relation positions.

Chapter 35 From the Parity Obstacle to Reflection Localization

Pure divisibility-survivor information often cannot distinguish different parities of multiplicative depth.

Once

$$\lambda = (-1)^{\Omega}$$

is incorporated into the state, multiplicative-depth parity can be tracked explicitly, and the odd channel can even complete prime certification at the cube-root layer.

Yet fixed sums still require

the correct two primes to lie in the same reflection fiber.

(35.1)

Thus the final geometric question for fixed sums is no longer

is an endpoint sufficiently prime-like?

but the pointwise coupling

concrete atomic position ↔ concrete reflected position ↔ concrete multiplicative responsibility. (35.2)

This is **reflection localization** for fixed sums.

Part VI Fixed Sums: A Terminal-Saturation Relation Sieve

Chapter 36 From the General Relation Sieve to Fixed Sums

Fix an even integer

$$N = 2m, \qquad m \geq 3.$$

In atomic coordinates, the relation is

$$\boxed{a + b + 1 = m.} \tag{36.1}$$

Define

$$a \boxplus b = a + b + 1.$$

The fixed-sum relation is a vertical slice of the binary atomic relation plane from Part III.

Here every target m has a special property:

> *its endpoints are automatically bounded above.*

This makes fixed sums the canonical example of a **terminal-saturation relation sieve**.

Chapter 37 Matrix-to-Subrelation Readout Inheritance

Let

$$\theta: \Gamma_\infty \to Y$$

be any relation coordinate.

Define

$$\Gamma^{(\theta=y)} = \theta^{-1}(y).$$

Because

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots,$$

if

$$r \in \Gamma_k^{(\theta=y)},$$

then for every

$$j \geq k,$$

we still have

$$r \in \Gamma_j^{(\theta=y)}.$$

Therefore

a subreadout inherits the parent relation;

a readout does not recreate a relation.

(37.1)

For fixed sums, simply take

$$\theta = \sigma.$$

Chapter 38 Why Fixed Sum Is One of the Lowest-Order Exact Relations

Two odd primes

$$2a+1, \qquad 2b+1$$

satisfy

$$(2a+1)+(2b+1)=2(a+b+1).$$

Therefore

$$\boxed{\Phi(a)+\Phi(b)=2(a \boxplus b).} \tag{38.1}$$

A fixed-sum relation has three simultaneous rigidities:

1. it has only two endpoints;
2. both endpoints require exact atomic depth 1;
3. the endpoints must lie in the same specified fixed-sum fiber.

It therefore cannot be determined from marginal prime counts alone.

Chapter 39 Finite History and Fixed-Sum Support

Write the atomic sequence as

$$A = \{a_1 < a_2 < a_3 < \cdots\},$$

and its finite history as

$$A_k = \{a_1, \ldots, a_k\}.$$

Define the finite binary atomic relation matrix

$$\boxed{\Gamma_k = \{\{a_i, a_j\}: 1 \le i \le j \le k\}.} \tag{39.1}$$

When the k-th atom is born, the new relation layer is

$$\boxed{\Lambda_k = \{\{a_i, a_k\}: 1 \le i \le k\}.} \tag{39.2}$$

Define the fixed-sum support

$$\boxed{S_k = A_k \boxplus A_k.} \tag{39.3}$$

The fixed-sum projection of the newly born relation row is

$$\boxed{R_k = a_k \boxplus A_k.} \tag{39.4}$$

Hence

$$\boxed{S_k = S_{k-1} \cup R_k.} \tag{39.5}$$

This is the atom-by-atom growth law of relation support.

Chapter 40 Translation of Relation Rows

Write

$$h_k = a_k - a_{k-1}.$$

The historical relation row of the preceding atom is

$$R_{k-1}^{\text{hist}} = a_{k-1} \boxplus A_{k-1}.$$

Then

$$\boxed{a_k \boxplus A_{k-1} = h_k + (a_{k-1} \boxplus A_{k-1}).} \tag{40.1}$$

That is,

the historical part of the new relation row is a rigid translation of the preceding relation row.

(40.2)

Adding the self-pairing

$$a_k \boxplus a_k = 2a_k + 1$$

gives the complete newly born fixed-sum row.

Chapter 41 Recurrence for Fixed-Sum Multiplicity

Define

$$\boxed{G_k(m) = \#\Gamma_k^{(\sigma=m)}.} \tag{41.1}$$

Then at the birth of the k-th atom,

$$\boxed{G_k(m) = G_{k-1}(m) + \mathbf{1}_{R_{k-1}^{\mathrm{hist}}}(m - h_k) + \mathbf{1}_{\{2a_k+1\}}(m).} \tag{41.2}$$

This recurrence preserves

$$\boxed{\text{historical accumulation} + \text{prime-gap translation} + \text{new self-pairing.}} \tag{41.3}$$

It is the exact update equation for the fixed-sum multiplicity field.

Chapter 42 Terminal-Horizon Operator

For a general relation coordinate

$$\theta: \Gamma_\infty \to Y,$$

define

$$\boxed{\tau_\theta(y) = \min\{k: \theta^{-1}(y) \cap \Gamma_\infty = \theta^{-1}(y) \cap \Gamma_k\},} \tag{42.1}$$

whenever the minimum exists.

This operator answers only

for target y, can any new relation information still appear in the future?

It does not answer whether the target fiber is nonempty.

Therefore

terminality does not create existence; it closes future change. (42.2)

Chapter 43 Natural Terminal Freezing for Fixed Sums

If

$$m \le a_{k+1} + 1,$$

then every future atom is at least

$$a_{k+1},$$

while the other atom is at least

$$a_1 = 1.$$

Hence the smallest fixed sum of any future new relation is

$$a_{k+1} + 2 > m.$$

Therefore

$$\boxed{m \le a_{k+1} + 1 \Rightarrow \Gamma_{\infty}^{(\sigma=m)} = \Gamma_{k}^{(\sigma=m)}.} \tag{43.1}$$

This is terminal freezing for fixed sums.

Chapter 44 Natural Terminal Segments

Define

$$\boxed{J_k = \{a_k + 2, \dots, a_{k+1} + 1\}.} \tag{44.1}$$

Theorem 44.1 Partition by Natural Terminal Segments

$$\boxed{\mathbb{Z}_{\geq 3} = \bigsqcup_{k \geq 1} J_k.} \tag{44.2}$$

Proof

The final point of the k-th segment is

$$a_{k+1} + 1.$$

The first point of the $(k+1)$-st segment is

$$a_{k+1} + 2.$$

The segments are adjacent and disjoint.

Moreover,

$$a_1 = 1,$$

so the first segment begins at 3.

The atomic sequence is unbounded, so these segments cover all of

$$\mathbb{Z}_{\geq 3}.$$

This proves the theorem.

Therefore

every fixed-sum target belongs to exactly one finite historical world.

(44.3)

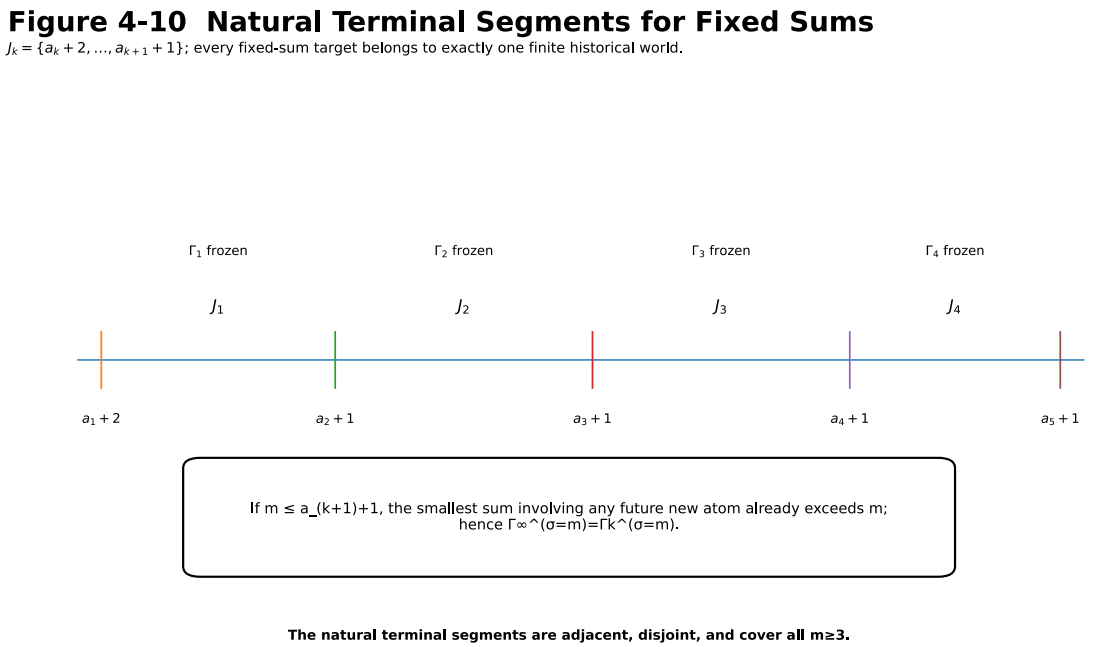


Figure 4-10. Natural Terminal Segments for Fixed Sums. The natural terminal segments $J_k = \{a_k + 2, \ldots, a_{k+1} + 1\}$ are adjacent, disjoint, and cover all fixed-sum targets $m \geq 3$. If $m \in J_k$, future atoms can no longer create a new relation with sum m, so the fiber freezes after layer k.

Chapter 45 Returning to the Even Axis: Prime Gaps Determine Terminal-Segment Width

Let

$$p_k = 2a_k + 1.$$

If

$$m \in J_k,$$

then

$$N = 2m$$

lies in the even segment

$$\boxed{E_k = \{p_k + 3, p_k + 5, \dots, p_{k+1} + 1\}.} \tag{45.1}$$

and

$$\boxed{\{6,8,10, \dots\} = \bigsqcup_{k\geq 1} E_k.} \tag{45.2}$$

The segment width satisfies

$$\boxed{|E_k| = a_{k+1} - a_k = \frac{p_{k+1} - p_k}{2}.} \tag{45.3}$$

Thus the gap between consecutive primes directly determines the number of newly terminalized targets.

Chapter 46 Defects and the First Defect

Define the relation defect in the k-th terminal segment by

$$\boxed{H_k = J_k \setminus S_k.} \tag{46.1}$$

If

$$H_k = \varnothing,$$

then

$$J_k \subseteq S_k.$$

That is, the terminal segment is completely occupied by fixed-sum relations.

If

$$H_k \neq \varnothing,$$

define the first defect

$$\boxed{F_k = \min H_k.} \tag{46.2}$$

A relation sieve does not ask

> *How many candidate integers remain?*

but rather

> **among relation targets that are already terminal, which targets still lack a genuine relation witness?**

(46.3)

Chapter 47 Reciprocal Motion Within a Fixed Sum

Take

$$m \in J_k.$$

Write uniquely

$$\boxed{m = a_k + 2 + r,} \tag{47.1}$$

where

$$0 \le r < a_{k+1} - a_k.$$

Define the historical distance cloud

$$\boxed{D_k = \{a_k - a_i : 1 \le i \le k\}.} \tag{47.2}$$

If

$$d = a_k - a_i,$$

then

$$a_i = a_k - d.$$

The requirement

$$m = a_i + a_j + 1$$

is equivalent to

$$a_j = r + d + 1.$$

Therefore

$$\boxed{m \in A_k \boxplus A_k \Leftrightarrow (r + 1 + D_k) \cap A_k \neq \varnothing.} \tag{47.3}$$

This is the moving-intersection equivalence.

It shows that

a fixed sum is not two independent searches; it consists of two reciprocal roles within the same sum relation.

(47.4)

Chapter 48 Finite Relation Certificates

Let

$$h_k = a_{k+1} - a_k.$$

For each

$$r = 0, \dots, h_k - 1,$$

suppose one chooses

$$d_r \in D_k$$

such that

$$r + d_r + 1 \in A_k.$$

Set

$$a_i = a_k - d_r,$$

$$a_j = r + d_r + 1.$$

Then

$$a_i + a_j + 1 = a_k + r + 2 = m.$$

The corresponding ordinary primes

$$p_i = 2a_i + 1, \qquad p_j = 2a_j + 1$$

satisfy

$$p_i + p_j = 2m.$$

Thus a finite certificate for an entire terminal segment can consist of a family

$$\boxed{\{d_r\}_{0 \le r < h_k}} \tag{48.1}$$

where each d_r supplies one concrete relation witness.

Part VII Exact Additive–Multiplicative Completion Objects

Chapter 49 Position Level: Complementary Identities of Atoms and Multiplicative Occupancy

In odd-number atomic coordinates, define

$$A = \{a: \Phi(a) \text{ is an odd prime}\},$$

$$\mathcal{M} = \{a: \Phi(a) \text{ is an odd composite}\}.$$

Then

$$\boxed{\mathbb{N}_{\geq 1} = A \sqcup \mathcal{M}.} \tag{49.1}$$

Define

$$\alpha(n) = \mathbf{1}_A(n),$$

$$\beta(n) = \mathbf{1}_{\mathcal{M}}(n).$$

Then

$$\boxed{\alpha(n) + \beta(n) = 1.} \tag{49.2}$$

Therefore

the final identity of a natural-number position has exactly two complementary readings: multiplicative occupancy or atomic remainder.

(49.3)

Chapter 50 Temporarily Unread States Within a Finite Horizon

At finite layer k, define

$$A_k = \{a_1, \dots, a_k\},$$

$$\mathcal{M}_k = \bigcup_{i \le k} \mathcal{R}_{a_i},$$

and

$$\boxed{U_k = \mathbb{N}_{\ge 1} \setminus (A_k \cup \mathcal{M}_k).} \tag{50.1}$$

A finite horizon therefore temporarily has three states:

$$A_k, \qquad \mathcal{M}_k, \qquad U_k.$$

But U_k is not a third final identity.

If

$$n < a_{k+1},$$

then

$$\boxed{U_k \cap [1, a_{k+1} - 1] = \varnothing.} \tag{50.2}$$

If

$$m \in J_k,$$

all endpoints of fixed-sum slots are less than a_{k+1}.

Therefore, at the natural terminal layer of that target,

$$\boxed{U_k \text{ has disappeared from every relevant endpoint.}} \tag{50.3}$$

Chapter 51 Fixed Sum as Re-Pairing Only

Fix

$$m \geq 3.$$

Define the complementary map

$$\boxed{\rho_m(a) = m - 1 - a.} \tag{51.1}$$

Then

$$a \boxplus \rho_m(a) = m.$$

This map does not alter endpoint identities.

It only

organizes two natural-number positions that already carry fixed identities into the same fixed-sum relation slot.

(51.2)

Therefore

addition does not recreate relations;

a fixed sum is a relational pairing readout of pre-existing endpoint identities.

(51.3)

Chapter 52 Relation-Slot Exhaustion Theorem

For any fixed sum

$$m \geq 3,$$

define all unordered relation slots

$$\boxed{\mathcal{S}_m = \{\{a, b\} : a + b + 1 = m,\ 1 \leq a \leq b\}.} \tag{52.1}$$

After terminality, every endpoint has only one of two identities:

$$A \quad \text{or} \quad \mathcal{M}.$$

Hence

$$\boxed{\mathcal{S}_m = \mathcal{S}_m^{AA} \sqcup \mathcal{S}_m^{A\mathcal{M}} \sqcup \mathcal{S}_m^{\mathcal{M}\mathcal{M}}.} \tag{52.2}$$

There is no fourth terminal relation identity.

This is relation-slot exhaustion for fixed sums.

Chapter 53 Exact Slotwise Completion Identity

Since

$$\hat{\beta}(a) = 1 - \hat{\alpha}(a), \qquad \hat{\beta}(b) = 1 - \hat{\alpha}(b),$$

we have, slot by slot,

$$\boxed{\hat{\alpha}(a)\hat{\alpha}(b) = \hat{\alpha}(a) + \hat{\alpha}(b) - 1 + \hat{\beta}(a)\hat{\beta}(b).} \tag{53.1}$$

Proof

$$\begin{aligned}\hat{\alpha}(a) + \hat{\alpha}(b) - 1 + \hat{\beta}(a)\hat{\beta}(b) &= \hat{\alpha}(a) + \hat{\alpha}(b) - 1 \\ &\quad +(1 - \hat{\alpha}(a))(1 - \hat{\alpha}(b)) \\ &= \hat{\alpha}(a)\hat{\alpha}(b).\end{aligned}$$

This proves the identity.

It expresses exactly that

double-atomic slots and double-multiplicative-occupancy slots are complementary readouts within the same fixed-sum slot decomposition.

(53.2)

Chapter 54 Conservation of the Entire Relation-Slot Set

The total number of unordered fixed-sum slots is

$$\boxed{H(m) = \left\lfloor \frac{m-1}{2} \right\rfloor.} \tag{54.1}$$

Define

$$G(m) = \#\mathcal{S}_m^{AA},$$

$$X(m) = \#\mathcal{S}_m^{A\mathcal{M}},$$

$$C(m) = \#\mathcal{S}_m^{\mathcal{M}\mathcal{M}}.$$

By relation-slot exhaustion,

$$\boxed{H(m) = G(m) + X(m) + C(m).} \tag{54.2}$$

If the numbers of atomic left and right endpoints are counted separately as

$$L(m), \qquad R(m),$$

then

$$L(m) + R(m) = 2G(m) + X(m).$$

Eliminating $X(m)$ gives

$$\boxed{G(m) = L(m) + R(m) - H(m) + C(m).} \tag{54.3}$$

This is an exact identity.

Figure 4-11 Exact Slotwise Completion

Every terminal fixed-sum slot belongs to exactly one of AA, AM, or MM; the slotwise identity and global conservation turn 'completion' into an exact ledger.

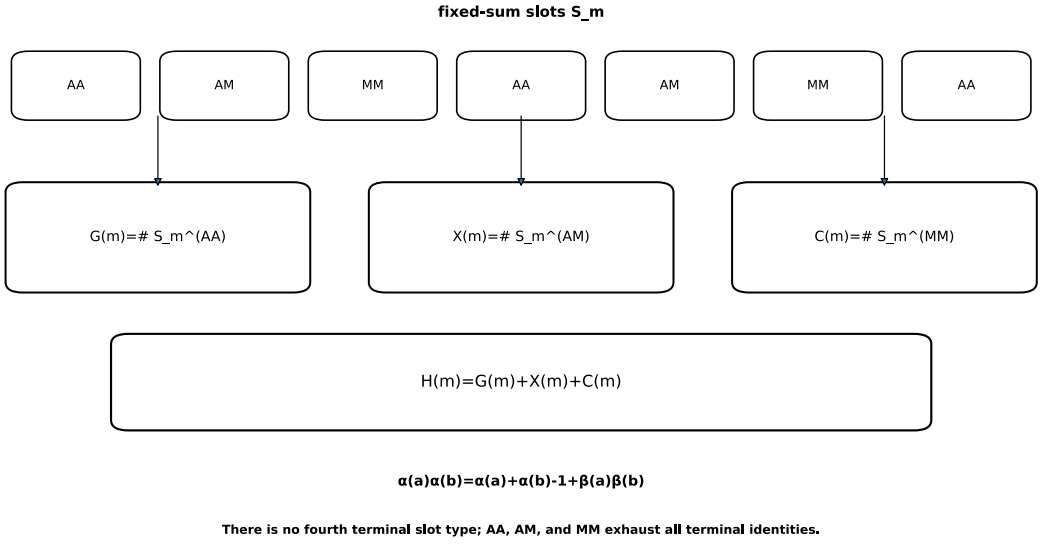


Figure 4-11. Exact Slotwise Completion. At the natural terminal layer of a fixed sum, every relation slot belongs to exactly one of AA, $A\mathcal{M}$, or $\mathcal{M}\mathcal{M}$, so $H(m) = G(m) + X(m) + C(m)$. The

slotwise identity $\hat{\alpha}(a)\hat{\alpha}(b) = \hat{\alpha}(a) + \hat{\alpha}(b) - 1 + \hat{\beta}(a)\hat{\beta}(b)$ places double-atomic and double-multiplicative slots inside one exact decomposition.

Chapter 55 Responsibility Decomposition of Double Multiplicative Occupancy

Every odd composite has a unique least prime factor.

Define

$$\boxed{E_p = \{a \in \mathcal{M} : P^-(2a+1) = p\}.} \tag{55.1}$$

Then

$$\boxed{\mathcal{M} = \bigsqcup_{p \in \mathbb{P}_{\text{odd}}} E_p.} \tag{55.2}$$

Hence double-multiplicative slots decompose further as

$$\boxed{C(m) = \sum_{p,q} C_{p,q}(m),} \tag{55.3}$$

where

$$C_{p,q}(m) = \#\{\{a,b\} \in S_m^{\mathcal{M}\mathcal{M}} : a \in E_p,\ b \in E_q\}.$$

If

$$C_{p,q}(m) > 0,$$

then

$$2a+1 = pr, \qquad 2b+1 = qs,$$

with

$$r \geq p, \qquad s \geq q.$$

From

$$(2a+1) + (2b+1) = 2m$$

we obtain

$$pr + qs = 2m.$$

Therefore

$$\boxed{C_{p,q}(m) > 0 \Rightarrow p^2 + q^2 \leq 2m.} \tag{55.4}$$

If

$$p = q,$$

then

$$p(r+s) = 2m,$$

and hence

$$\boxed{C_{p,p}(m) > 0 \Rightarrow p \mid m.} \tag{55.5}$$

A double-multiplicative slot is not an anonymous “failed relation”; it retains a clear structure of multiplicative responsibility.

Chapter 56 Temporal Offset Between Atomic Birth and Exclusive Multiplicative Action

Let

$$p = 2a + 1$$

be a new atom.

At birth it immediately forms new relations with the historical atoms.

But its first appearance as the least prime factor of a composite occurs only at

$$p^2.$$

In atomic coordinates,

$$\boxed{\Phi^{-1}(p^2) = 2a(a+1).} \tag{56.1}$$

Therefore

the birth of atomic identity precedes its exclusive action on future multiplicative occupancy.

(56.2)

Relation supply and multiplicative feedback possess a natural temporal offset.

Chapter 57 Concrete Relation Slots: $N = 12$

Take

$$N = 12, \qquad m = 6.$$

The unordered atomic-coordinate slots are

$$(1,4), \qquad (2,3).$$

Since

$$4 = 1 \star 1 \in \mathcal{M},$$

we have

$$(1,4) \in A\mathcal{M}.$$

But

$$2,3 \in A,$$

so

$$(2,3) \in AA.$$

The ordinary-integer readout is

$$3 + 9 = 12$$

and

$$\boxed{5 + 7 = 12.}$$

There is no process of "first failing and then being filled by another pair."

More accurately,

a fixed sum contains slots of different identities from the outset; the target reads only the AA slo (57.1)

Chapter 58 Concrete Relation Slots: $N = 16, N = 18, N = 20, N = 40$

The $N = 12$ example in Chapter 57 showed that

$$A\mathcal{M} \quad \text{and} \quad AA$$

can coexist in the same fixed sum.

The following four targets further show how the relation alphabet unfolds in concrete fibers.

58.1 $N = 16$

Here

$$m = 8.$$

The unordered atomic-coordinate slots are

$$(1,6), \quad (2,5), \quad (3,4).$$

The corresponding ordinary odd endpoints are

$$(3,13), \quad (5,11), \quad (7,9).$$

Their identities are, respectively,

$$\boxed{AA, \quad AA, \quad A\mathcal{M}.}$$

Thus

$$\boxed{16 = 3 + 13 = 5 + 11.}$$

The same fixed-sum fiber contains two AA witnesses.

Therefore fixed sums can read not only

$$G(m) > 0,$$

but also representation multiplicity.

58.2 $N = 18$

Here

$$m = 9.$$

The unordered slots are

$$(1,7), \quad (2,6), \quad (3,5), \quad (4,4).$$

The corresponding ordinary endpoints are

$$(3,15), \quad (5,13), \quad (7,11), \quad (9,9).$$

Thus

$$(1,7) \in A\mathcal{M},$$

$$(2,6), (3,5) \in AA,$$

and

$$(4,4) \in \mathcal{M}\mathcal{M}.$$

A single fixed-sum fiber therefore realizes the entire coarse relation alphabet

$$\boxed{\Sigma_{\text{sum}} = \{AA, A\mathcal{M}, \mathcal{M}\mathcal{M}\}.}$$

Its AA-readout gives

$$\boxed{18 = 5 + 13 = 7 + 11.}$$

This example shows that

$$\boxed{\text{Goldbach reads the } AA \text{ word inside the same complete relation alphabet.}} \tag{58.1}$$

58.3 $N = 20$

Here

$$m = 10.$$

The fixed-sum slots are

$$(1,8), \quad (2,7), \quad (3,6), \quad (4,5).$$

Their identities are

$$AA, \quad A\mathcal{M}, \quad AA, \quad \mathcal{M}A.$$

Therefore

$$20 = 3 + 17 = 7 + 13.$$

58.4 $N = 40$

The relation

$$15 + 25 = 40$$

has two multiplicatively occupied endpoints:

$$15 = 3 \cdot 5, \qquad 25 = 5^2.$$

Thus it is an

$$\mathcal{M}\mathcal{M}$$

slot.

This shows that the third terminal relation type genuinely occurs, but it is not a third endpoint identity.

It is simply the relation readout of two multiplicatively occupied endpoints under the same sum.

Part VIII Terminal Saturation, Universal Readout, and the Standard Toolset

Chapter 59 Goldbach Relation Completeness and Its Exact Position

Define

$$\boxed{G(m) = \#\mathcal{S}_m^{AA}.} \tag{59.1}$$

Then

$$\boxed{G(m) > 0 \Leftrightarrow \Gamma_\infty^{(\sigma=m)} \neq \varnothing.} \tag{59.2}$$

The odd-prime portion of the classical binary Goldbach conjecture is equivalent to

$$\boxed{\forall m \geq 3, \qquad G(m) > 0.} \tag{59.3}$$

It is also equivalent to

$$\boxed{\sigma(\Gamma_\infty) = \mathbb{Z}_{\geq 3}.} \tag{59.4}$$

and to

$$\boxed{A \boxplus A = \mathbb{Z}_{\geq 3}.} \tag{59.5}$$

We call

$$\boxed{\sigma(\Gamma_\infty) = \mathbb{Z}_{\geq 3}}$$

the **relation completeness** of the fixed-sum relation.

It means that every legal fixed-sum target is read by at least one AA relation.

Theorem 59.1 Equivalence of Relation Completeness and Universal Goldbach

The following three statements are equivalent:

$$\boxed{\forall m \geq 3, \quad G(m) > 0,} \tag{59.6}$$

$$\boxed{\sigma(\Gamma_\infty) = \mathbb{Z}_{\geq 3},} \tag{59.7}$$

and

$$\boxed{A \boxplus A = \mathbb{Z}_{\geq 3}.} \tag{59.8}$$

Proof

By definition,

$$G(m) > 0 \Leftrightarrow \Gamma_\infty^{(\sigma=m)} \neq \varnothing.$$

Thus

$$\forall m \geq 3,\ G(m) > 0$$

if and only if

$$\sigma(\Gamma_\infty) = \mathbb{Z}_{\geq 3}.$$

On the other hand,

$$\sigma(\{a, b\}) = a \boxplus b.$$

Hence

$$\sigma(\Gamma_\infty) = A \boxplus A.$$

The three statements are therefore equivalent. This proves the theorem.

Consequently,

relation-level surjectivity is not an additional completion mechanism; it is exactly the universal nonemptiness readout of the fixed-sum AA slots. (59.9)

The case

$$N = 4$$

is handled separately by

$$2 + 2.$$

Therefore

Goldbach is the universal nonemptiness readout of the AA slots in the exact additive–multiplicat (59.10)

This identifies the exact position of the proposition in the unified matrix. Together with the natural terminality, relation-slot exhaustion, exact additive–multiplicative completion, and relation completeness established in this part, the universal nonemptiness readout of Goldbach is closed here.

Chapter 60 The Threefold Terminal Equivalence

If

$$m \in J_k,$$

then

$$\Gamma_{\infty}^{(\sigma=m)} = \Gamma_k^{(\sigma=m)}.$$

Therefore the following three statements are equivalent:

$$\boxed{\Gamma_{\infty}^{(\sigma=m)} \neq \varnothing,}$$

$$\boxed{m \in A \boxplus A,}$$

$$\boxed{m \in A_k \boxplus A_k.}$$

Hence the universal proposition is equivalent to

$$\boxed{\forall k, \qquad J_k \subseteq A_k \boxplus A_k.} \tag{60.1}$$

Thus

$$\boxed{\text{classical Goldbach} \Leftrightarrow A \boxplus A = \mathbb{Z}_{\geq 3} \Leftrightarrow \forall k,\ J_k \subseteq A_k \boxplus A_k.} \tag{60.2}$$

This is the exact terminalization of the universal fixed-sum proposition.

Chapter 61 Readout–Terminal Placement Theorem

Let

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_\infty$$

be a persistent relation system, and let

$$\theta : \Gamma_\infty \to Y$$

be a relation coordinate.

Suppose the target space is partitioned as

$$\boxed{Y = \bigsqcup_{k\geq 1} J_k,} \tag{61.1}$$

and

$$y \in J_k \Rightarrow \Gamma_\infty^{(\theta=y)} = \Gamma_k^{(\theta=y)}.$$

Then

$$\boxed{y \in \theta(\Gamma_\infty) \Leftrightarrow y \in \theta(\Gamma_k) \qquad (y \in J_k).} \tag{61.2}$$

This is **readout–terminal placement**.

It expresses that

terminality does not create global existence;

it places relation facts already present in the matrix exactly into a finite historical layer.

(61.3)

Chapter 62 Complete Finite Sections

Take the natural terminal segment

$$J_k = \{a_k + 2, \dots, a_{k+1} + 1\},$$

and let

$$h_k = a_{k+1} - a_k.$$

Every target can be written uniquely as

$$m = a_k + 2 + r, \qquad 0 \le r < h_k.$$

Define

$$\boxed{\mathfrak{G}_k = \left\{(r,d): \begin{matrix} 0 \le r < h_k, \\ d \in D_k, \\ r + d + 1 \in A_k \end{matrix}\right\}.} \tag{62.1}$$

The fiber at fixed r is

$$\boxed{\mathfrak{G}_k(r) = \{d: (r,d) \in \mathfrak{G}_k\}.} \tag{62.2}$$

By the moving-intersection theorem,

$$\boxed{m = a_k + 2 + r \text{ has a binary atomic representation} \Leftrightarrow \mathfrak{G}_k(r) \neq \varnothing.} \tag{62.3}$$

Thus

$$\boxed{\text{exact readout} = \text{moving intersection} = \text{nonemptiness of the fixed-sum relation section.}} \tag{62.4}$$

Figure 4-12 Goldbach Terminal-Saturation Closure

Main proof chain of Part IV: every fixed-sum target enters a finite natural terminal segment; complete finite sections and relation completeness yield the universal nonempty readout.

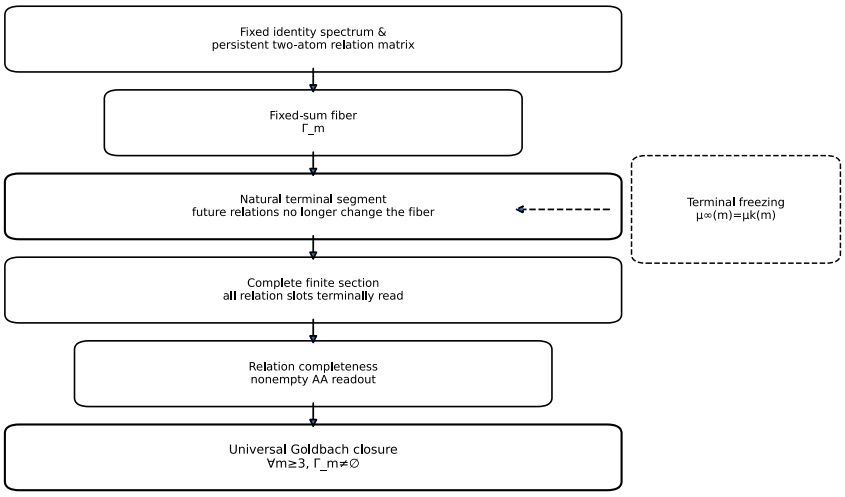


Figure 4-12. Goldbach Terminal-Saturation Closure. Once a fixed-sum fiber enters its natural terminal segment, future relations no longer change it. The complete finite section and relation completeness meet on the same terminal segmentation to yield Goldbach's universal nonempty readout.

Chapter 63 Support and Multiplicity

A relation sieve does not merely decide

present or absent.

It also preserves representation multiplicity.

Define

$$\boxed{\mu_k(m) = \#\Gamma_k^{(\sigma=m)}.} \tag{63.1}$$

For a terminal target

$$m \in J_k,$$

we have

$$\boxed{\mu_\infty(m) = \mu_k(m).} \tag{63.2}$$

Therefore

$$m \in S_k \Leftrightarrow \mu_k(m) \geq 1.$$

One may further study the distribution of

$$\mu_k(m) = 1,2,3, \ldots.$$

This connects naturally with the result in Part III that Green–Tao chains feed high multiplicities into fixed sums.

Chapter 64 Eight Standard Relation-Sieve Operators

Let

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$

be a persistent relation system and let

$$\theta$$

be a relation coordinate.

Define the following eight standard objects.

64.1 Birth Operator

$$\boxed{\mathsf{B}_k = \Gamma_k \backslash \Gamma_{k-1} = \Lambda_k.} \tag{64.1}$$

It records the relations newly born at layer k.

64.2 Projection Operator

$$\boxed{\mathsf{P}_\theta(\Gamma_k) = S_k^\theta.} \tag{64.2}$$

It projects the complete relation ledger onto the target coordinate.

64.3 Terminal Operator

$$\boxed{\tau_\theta(y) = \min\left\{k : \Gamma_\infty^{(\theta=y)} = \Gamma_k^{(\theta=y)}\right\}.} \tag{64.3}$$

64.4 Defect Operator

$$\boxed{\mathsf{H}_k = J_k \backslash S_k^\theta.} \tag{64.4}$$

64.5 First-Defect Operator

If

$$\mathsf{H}_k \neq \varnothing,$$

define

$$\boxed{\mathsf{F}_k = \min \mathsf{H}_k.} \tag{64.5}$$

64.6 Opportunity Operator

For the first defect F_k, define the finite relation set still capable of changing its status by

$$\boxed{\mathsf{O}(F_k) = \mathcal{H}_k^F.} \tag{64.6}$$

It compresses an infinite narrative into a finite set of possibilities.

64.7 Certificate Operator

If a target is occupied, choose an actual relation

$$\boxed{\mathsf{C}(y) = r_y \in \Gamma^{(\theta=y)}.} \tag{64.7}$$

For fixed sums, one can record concretely

$$(r, d_r, a_i, a_j, p_i, p_j).$$

64.8 Saturation Operator

Define

$$\boxed{\mathsf{Sat}_\theta(k) = \mathbf{1}_{\{J_k \subseteq S_k^\theta\}}.} \tag{64.8}$$

If

$$\mathsf{Sat}_\theta(k) = 1 \qquad \forall k,$$

then the entire target space is terminally saturated.

The eight objects form the chain

$$\boxed{\mathsf{B} \to \mathsf{P} \to \tau \to \mathsf{H} \to \mathsf{F} \to \mathsf{O} \to \mathsf{C} \to \mathsf{Sat}.} \tag{64.9}$$

Figure 4-13 Eight Standard Relation-Sieve Operators
From relation birth to segment saturation, eight operators compress the relation sieve into a reusable and software-ready standard chain.

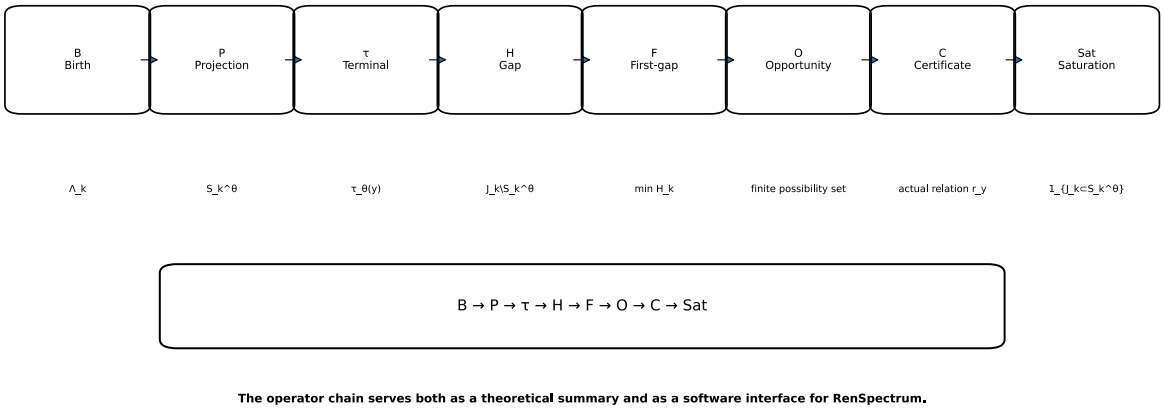


Figure 4-13. Eight Standard Relation-Sieve Operators. Part IV compresses the relation sieve into eight standard operators: Birth B, Projection P, Terminal τ, Gap H, First-gap F, Opportunity O, Certificate C, and Saturation Sat. They form the standard chain $\mathsf{B} \to \mathsf{P} \to \tau \to \mathsf{H} \to \mathsf{F} \to \mathsf{O} \to \mathsf{C} \to \mathsf{Sat}$.

Chapter 65 Terminal-Saturation Type and Persistent Non-Extinction Type

Relation sieves have at least two basic forms.

65.1 Terminal-Saturation Type

Every target y satisfies

$$\tau_\theta(y) < \infty.$$

The global quantifier is

$$\forall y.$$

The characteristic structure is

$$\boxed{\text{target segmentation} + \text{finite terminality} + \text{segmentwise saturation.}} \tag{65.1}$$

Fixed sums belong to this class.

65.2 Persistent Non-Extinction Type

The entire relation fiber has no common finite terminal horizon.

Each concrete instance is still finitely certified, but globally one requires

$$\exists^{\infty}$$

or

$$\forall K\ \exists k > K.$$

The characteristic structure is

$$\boxed{\text{relation supply} + \text{finite absorption} + \text{persistence across horizons.}} \tag{65.2}$$

Infinite recurrence of a fixed difference belongs to this class.

Therefore

both use relation sieves; the difference lies in the outer quantifier and terminal structure.

(65.3)

Chapter 66 Why Fixed Sum and Fixed Difference Cannot Use the Same Terminal Logic

A fixed sum

$$a_i + a_j + 1 = m$$

automatically bounds its endpoints, so

$$\tau_\sigma(m) < \infty.$$

A fixed difference

$$a_j - a_i = h$$

does not prevent the relation as a whole from moving arbitrarily far outward.

Hence a global fixed-difference fiber generally has no common finite terminal horizon.

Therefore

fixed sum: every target has a finite terminal horizon; fixed difference: every instance is finitely certified, but the whole fiber must persist.

(66.1)

This determines a different mathematical task for persistent-type relation sieves.

Part IX Capability Boundaries and General Interfaces of Relation Sieves

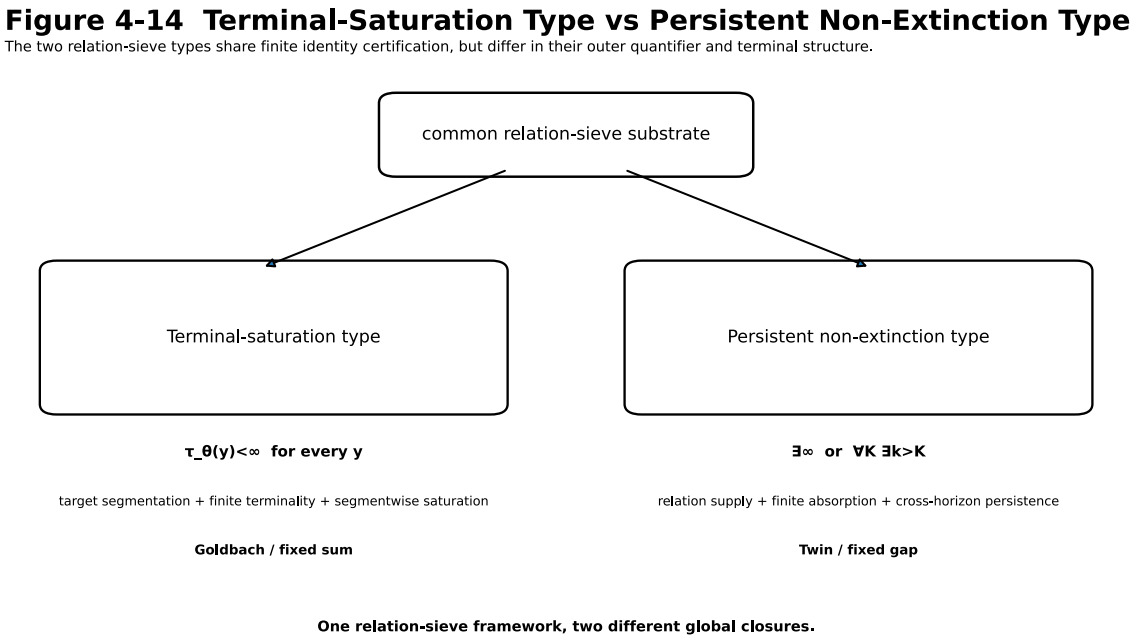


Figure 4-14. Terminal-Saturation Type vs Persistent Non-Extinction Type. Relation sieves have at least two basic forms. The terminal-saturation type requires every target y to have finite terminality $\tau_\theta(y) < \infty$ and then saturates the target space segment by segment; fixed sums are of this type. The persistent non-extinction type has no common finite terminal horizon for the whole fiber; instead, individual instances are finitely certified while the cross-horizon family satisfies $\exists^\infty$ or an equivalent persistence quantifier. Infinite recurrence of a fixed gap is of this type.

Chapter 67 Division of Labor Between Finite Terminality and Universal Closure

For a fixed target, the relation sieve reads exactly at the terminal layer whether

$$G(m) = 0 \quad \text{or} \quad G(m) > 0.$$

The universal proposition is then closed on the same terminal segmentation by relation completeness. Prime certification in the odd channel belongs to the endpoint-identity layer; positivity of fixed sums belongs to the reflection-relation layer. The two layers have distinct roles and are connected in the Goldbach proof chain of this part by the natural terminal segments and relation completeness.

Therefore

terminal certification resolves the truth value for a single target;

relation completeness together with natural terminal segmentation completes Goldbach's universal closure.

(67.1)

This agrees exactly with the quantifier architecture for batch proofs in Part III.

Chapter 68 Simple Cardinality Capacity Cannot Replace Reflection Localization

If a fixed sum failed, every potential one-end atom would have to be reflected into a multiplicatively occupied position.

Merely comparing

the number of atoms and the number of multiplicatively occupied positions

usually produces no contradiction.

The multiplicatively occupied set may be larger than the atomic set.

Likewise, crude capacities of residue classes associated with individual responsible primes may sum to far more than the number of atoms.

The quantity that must be controlled is the pointwise coupling

$$\boxed{\text{concrete atomic position} \to \text{concrete reflected position} \to \text{concrete first killer.}} \tag{68.1}$$

Chapter 69 Four Reusable Interfaces of the Relation-Faithful Sieve

69.1 Relation-Fidelity Interface

Preserve

$$\boxed{\text{position} + \text{endpoints} + \text{identity} + \text{death history}.}$$

This applies to all finite relation slots.

69.2 Atomic-Birth / Relation-Row Interface

After a new atom is born, the historical portion of its relation row is translated by the prime gap:

$$\boxed{a_k \boxplus A_{k-1} = h_k + (a_{k-1} \boxplus A_{k-1}).}$$

69.3 Multiplicative-Depth Channel Interface

Use

$$\Omega, \qquad \lambda = (-1)^{\Omega}$$

to record shallow multiplicative depth and parity transfer.

69.4 Failure-Certificate Interface

Failure of a finite all-prime-type relation can be recorded as

$$\boxed{\text{specific failed endpoint} + \text{unique least-prime-factor responsibility} + \text{relation position}.}$$

This lifts “failure” from a Boolean value to an auditable finite object.

Chapter 70 How a New Relation Enters Finite-Control Geometry

Any new relation can enter the relation sieve through the following sequence.

First, write the relation map

$$F = (F_1, \dots, F_r).$$

Second, specify the target depth set

$$B \subseteq \mathbb{N}^r.$$

Third, determine the finite observation region.

Fourth, determine the local obstructions

$$B_\ell(F).$$

Fifth, check local admissibility.

Sixth, determine the genuine-death rule.

Seventh, construct either a finite terminal horizon or a persistent horizon structure.

Eighth, preserve relation witnesses and failure certificates.

Ninth, invoke the corresponding quantifier-closure mechanism from Part III to complete batch closure.

Thus

$$\boxed{\text{relation orbit} \to \text{multiplicative depth} \to \text{local obstruction} \to \text{genuine death} \to \text{relation fidelity} \to} \tag{70.1}$$

Chapter 71 Table of Fundamental Theorems in This Part

Fundamental Theorem AA Atomic-Vertex Preimage

$$\boxed{\{t: F_j(t) \text{ are all prime}\} = F^{-1}(\mathcal{V}_r).}$$

Fundamental Theorem AB Local Admissibility of Finite Patterns

$$\boxed{\nu_H(q) = q}$$

means that one local prime layer completely covers the pattern. Complete coverage need only be checked for

$$q \le |H|.$$

Fundamental Theorem AC CRT Survival Tree

$$\boxed{|A_s(H)| = \prod_{i=1}^{s}(p_i - \nu_H(p_i)).}$$

and

$$\boxed{\frac{|A_s(H)|}{W_s} = \prod_{i=1}^{s}\left(1 - \frac{\nu_H(p_i)}{p_i}\right).}$$

Fundamental Theorem AD Conservation of First Local Elimination

Let

$$\eta_i = 1 - \frac{\nu_H(p_i)}{p_i}.$$

Then

$$\boxed{1 = \prod_{i=1}^{s}\eta_i + \sum_{i=1}^{s}(1-\eta_i)\prod_{j<i}\eta_j.}$$

Fundamental Theorem AE Uniqueness of Genuine Death

$$\boxed{\kappa(n) = P^{-}(n)}$$

ensures that every odd composite genuinely dies exactly once.

Fundamental Theorem AF Relation First Killer

Every failed finite candidate relation of all-prime type has a unique earliest responsible prime.

Fundamental Theorem AG Relation Alphabet

$$\boxed{\omega(R) = (\iota(n_1), \dots, \iota(n_r)).}$$

The collection of all possible terminal identity words of a relation family is

$$\boxed{\Sigma_{\mathcal{R}}.}$$

Fundamental Theorem AH Cube-Root Certification of the Odd Channel

$$\boxed{z \geq X^{1/3} \Rightarrow O_z = \mathbf{1}_{\mathbb{P}} \quad \text{on } [3, X].}$$

Fundamental Theorem AI Odd-Channel Decomposition at $X^{1/4}$

$$\boxed{O_z = P_1 \sqcup P_3 \qquad (z \geq X^{1/4}).}$$

Fundamental Theorem AJ Terminal Freezing of Fixed Sums

$$\boxed{m \in J_k \Rightarrow \Gamma_{\infty}^{(\sigma=m)} = \Gamma_k^{(\sigma=m)}.}$$

Fundamental Theorem AK Partition by Natural Terminal Segments

$$\boxed{\mathbb{Z}_{\geq 3} = \bigsqcup_{k \geq 1} J_k.}$$

Fundamental Theorem AL Moving-Intersection Equivalence

$$\boxed{m = a_k + 2 + r \in A_k \boxplus A_k \Leftrightarrow (r + 1 + D_k) \cap A_k \neq \varnothing.}$$

Fundamental Theorem AM Relation-Slot Exhaustion

$$\boxed{\mathcal{S}_m = \mathcal{S}_m^{AA} \sqcup \mathcal{S}_m^{A\mathcal{M}} \sqcup \mathcal{S}_m^{\mathcal{M}\mathcal{M}}.}$$

Fundamental Theorem AN Exact Slotwise Completion

$$\boxed{\hat{\alpha}(a)\hat{\alpha}(b) = \hat{\alpha}(a) + \hat{\alpha}(b) - 1 + \hat{\beta}(a)\hat{\beta}(b).}$$

Fundamental Theorem AO Conservation of the Entire Relation-Slot Set

$$\boxed{H(m) = G(m) + X(m) + C(m).}$$

Fundamental Theorem AP Decomposition by Double-Multiplicative Responsibility

$$\boxed{\mathcal{M} = \bigsqcup_p E_p, \qquad C(m) = \sum_{p,q} C_{p,q}(m).}$$

Fundamental Theorem AQ Equivalence of Relation Completeness and Universal Goldbach

$$\boxed{\forall m \geq 3,\ G(m) > 0 \Leftrightarrow \sigma(\Gamma_\infty) = \mathbb{Z}_{\geq 3} \Leftrightarrow A \boxplus A = \mathbb{Z}_{\geq 3}.}$$

Fundamental Theorem AR Readout–Terminal Placement

If a target $y \in J_k$ is terminal at layer k, then

$$\boxed{y \in \theta(\Gamma_\infty) \Leftrightarrow y \in \theta(\Gamma_k).}$$

Figure 4-15 Dependency Map of the Foundational Theorems in Part IV

AA–AR are grouped into four consecutive modules; module labels and theorem tags are separated to keep the dependency map readable.

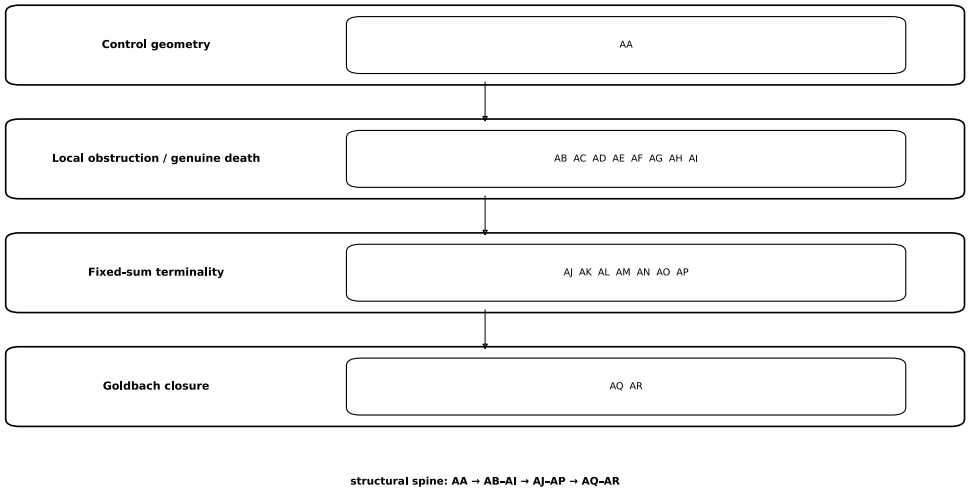


Figure 4-15. Dependency Map of the Foundational Theorems in Part IV. The foundational theorems AA–AR of Part IV fall, following the exposition, into four consecutive modules: control geometry via the atomic-vertex preimage; local admissibility, CRT, first elimination, genuine death, relation alphabets, and odd-channel certification; fixed-sum terminal freezing, natural terminal segments, moving intersections, slot exhaustion, and responsibility decomposition; and finally relation-completeness/Goldbach universal equivalence together with readout-terminal placement.

Conclusion What Does a Relation Sieve Actually Sieve?

From the viewpoint of an integer sieve, the action of a prime layer may appear simple:

$$q \mid n \quad \Rightarrow \quad n \text{ is removed.}$$

A relation problem is never that simple.

A relation must preserve

which position, paired with which position, at which layer it first fails, and when every future possibility has been clo

A relation sieve therefore does not truly sieve “integers.”

It sieves

the complete identity of a relation instance within finite multiplicative history.

This part first placed every finite relation in the same multiplicative-depth field

$$\Delta_r.$$

The all-prime relations form only its atomic vertex skeleton

$$\mathcal{V}_r.$$

Then every prime control layer acts on the relation orbit through local residue classes.

If one layer covers the whole parameter space, the relation is locally obstructed;

if no layer completely covers it, one obtains only local admissibility, which must not be rewritten as global occupancy.

Finite local phases are then unfolded completely by the CRT survival tree.

Every surviving parent phase produces p_i child phases at a new prime layer, of which

$$v_H(p_i)$$

continue to survive, while

$$v_H(p_i)$$

lose local admissibility for the first time at that layer.

Thus

$$\boxed{\text{all parameter phases} = \text{joint survival} + \text{first local eliminations at each layer.}}$$

Local sieving is therefore no longer merely a final product; it is a finite history traceable layer by layer.

Genuine death then turns ordinary “divisibility hits” into unique integer-identity histories:

every odd composite genuinely dies exactly once, at the layer of its least prime factor.

The relation first killer lifts this uniqueness to relation objects.

The relation alphabet answers a further question:

once a relation reaches terminality, what can it ultimately become?

The coarse fixed-sum alphabet is

$$AA, \qquad A\mathcal{M}, \qquad \mathcal{M}\mathcal{M}.$$

Genuine death tells us how a relation leaves the candidate layer;

the relation alphabet tells us which terminal identity type it ultimately occupies.

The odd–even dual channel shows that multiplicative death also carries multiplicative-depth information; cube-root certification shows that preserving richer genuine identity data can indeed move prime certification from the square-root scale to a shallower scale.

But the fixed-sum example simultaneously teaches that

stronger endpoint certification does not mean that the relation automatically holds.

A fixed sum ultimately reads

$$\langle f, J_N f \rangle,$$

a concrete reflection pairing.

The highest discipline of a relation sieve is therefore not "compress as much as possible," but

the most valuable role of a tool is to preserve the relation, not to replace it.

For fixed sums, this preservation ultimately yields the natural terminal segments

$$J_k = \{a_k + 2, \dots, a_{k+1} + 1\}.$$

Each target belongs to exactly one finite historical world.

After terminality, relation slots are exhausted into

$$AA, \qquad A\mathcal{M}/\mathcal{M}A, \qquad \mathcal{M}\mathcal{M}.$$

This gives the most important object-level sentence of this part:

multiplicative occupancy and atomic remainder are complementary identities of the same natural-number matrix.

and

multiplication does not compete with primes;
primes are precisely the exact remainder after multiplicative occupancy.
Addition does not recreate relations either;
it merely re-pairs these complementary identities along a fixed sum and reads them.

Thus the exact position of the fixed-sum proposition is

Goldbach is the universal nonemptiness readout of the AA slots in the exact additive–multiplicative completion object.

This does not erase the relation problem.

On the contrary, only by reconstructing fixed-sum reflection positions, natural terminality, relation-slot identities, genuine death, and finite certificates can the proposition be placed exactly inside the unified relation matrix.

Part IV therefore completes the chain

relation orbit
⇓
multiplicative-depth field
⇓
local obstruction and admissibility
⇓
CRT survival tree
⇓
first local elimination
⇓
genuine death and first killer
⇓
relation alphabet and relation-faithful sieve
⇓
terminal horizon
⇓
terminal relation geometry
⇓
universal Goldbach saturation closure.

By contrast, infinite recurrence of a fixed difference has no common terminal target segment.

It requires another type of relation sieve:

persistent non-extinction type.

There the decisive quantity is no longer “when does this target freeze?” but

whether relation supply continues to exceed finite absorption across infinitely many moving horizons.

Part V formulates this persistent structure as a supply–absorption ledger and cumulative net potential, and completes the infinite-recurrence closure for Twin.

Part V The Exact Architecture of Infinite Relations

— From Square Horizons, Relation Supply, and Finite Absorption to Cumulative Certificates and Global Quantifiers

Central Theme of This Part

Finite relations can be terminally certified, but infinite relations do not follow automatically from "computing larger and larger finite ranges."

Infinite quantifiers require mathematical objects of their own.

For relations of the infinite-recurrence type, the core issue is not whether one particular window succeeds, but rather a comparison, on the same ledger, between persistent relation supply and the maximal absorption capacity of the historical multiplicative structure:

Finite windows continually generate relation supply;the absorption capacity of the historical multiplicative structure is finite

Accordingly, this part expresses an infinite relation as a chain that can be verified layer by layer:

finite horizon → relation supply → absorption resources → maximal absorption capacity → net relation potential → cumula

Part IV has already written the internal control history of a finite relation as:

local admissibility → CRT survival tree → first local elimination → genuine death → relation alphabet.

Part V continues upward from this finite ledger: the CRT survival tree gives the admissible phase branches; B_H reads the actual relation supply generated by those admissible branches within a concrete square horizon; T_H and D_H then decompose that supply according to terminal identity; and the cumulative certificate finally enters the quantifier layer exactly matched to the proposition.

Every step is a finite object; infinity appears only in the final test of "unboundedness."

This constitutes the standard interface for infinite prime relations.

Abstract

Part IV established the first basic form of a relation sieve: the **terminal-saturation relation sieve**. A fixed-sum target has a finite terminal segment; each target freezes within a finite history, and relation completeness yields the universally nonempty readout of the terminal fiber.

This part establishes the second basic form: the **persistent non-extinction relation sieve**.

A relation of this type has no common finite terminal horizon for its entire fiber. The typical example is a fixed-difference relation:

$$p, \qquad p + d.$$

Each individual candidate can be completely certified within a finite square horizon, while the relation as a whole can move arbitrarily far to the right. Hence the global proposition is no longer necessarily of the form

$$\forall \theta \; \exists x,$$

but may instead be

$$\exists^{\infty} x.$$

The finite-control chain of Part IV acquires a direct numerical interface here. For a fixed pattern H, the CRT survival tree gives the phase branches allowed by the old control layers; restricting those branches to a concrete square horizon and requiring all endpoints to escape the old history gives the actual relation supply

$$\boxed{\mathcal{B}_H(r,s), \qquad B_H(r,s) = |\mathcal{B}_H(r,s)|.}$$

Thus

$$\boxed{\text{CRT survival tree} \to \text{actual horizon supply } B_H \to \text{terminal identity words} \to T_H + D_H.}$$

Local phase capacity and actual relation supply are thereby separated rigorously: the former records which residue phases remain admissible; the latter records how many complete relations those admissible phases actually generate in the current integer horizon.

This part first distinguishes seven principal global interfaces:

$$\boxed{\begin{array}{l}\text{INF_RECURRENCE,}\\ \text{UNIVERSAL_FIBER,}\\ \text{WINDOW_UNIVERSAL,}\\ \text{ARBITRARILY_LONG,}\\ \text{PARAMETER_INFINITE,}\\ \text{UNIVERSAL_PREDICATE,}\\ \text{BLOCK_SEPARATION.}\end{array}}$$

For an infinite-recurrence relation

$$R(x) \in \{0,1\},$$

define the cumulative count

$$C_R(X) = \sum_{x \le X} R\,(x).$$

Then

$$\boxed{\exists^{\infty} x\colon R(x) = 1 \Leftrightarrow C_R(X) \text{ is unbounded.}}$$

Thus the essence of infinite recurrence is not that "the horizon is large," but that the **cumulative number of relations has no upper bound**.

Next, this part establishes a unified square horizon for a fixed offset pattern

$$H = \{h_1, \dots, h_k\}.$$

Let

$$p < r < s$$

be consecutive odd primes, and define the interior relation window

$$\boxed{I_H(r,s) = \{n \in 2\mathbb{Z}+1 \colon r^2 < n + h < s^2 \text{ for every } h \in H\}.}$$

At the old-history level, define the relation supply

$$\boxed{B_H(r,s) = \#\{n \in I_H(r,s) \colon P^-(n+h) \ge r \text{ for every } h \in H\}.}$$

Define the true all-atomic relation count

$$\boxed{T_H(r,s) = \#\{n \in I_H(r,s) : n + h \in \mathbb{P} \text{ for every } h \in H\}.}$$

The number of deaths is

$$\boxed{D_H(r,s) = B_H(r,s) - T_H(r,s).}$$

When $r \geq 5$, every endpoint that survives the old history but ultimately dies must have the form

$$n + h_j = ru,$$

where

$$u \in \mathbb{P}, \qquad s \leq u < \frac{s^2}{r}.$$

Accordingly, define the finite absorption resource pool

$$\boxed{U(r,s) = \left\{u \in \mathbb{P} : s \leq u < \frac{s^2}{r}\right\}.}$$

A single resource u may absorb different relations through different endpoint roles in the pattern. To control this reuse, choose a finite wheel

$$Q$$

and set

$$M_Q = \prod_{q \in Q} q.$$

Define

$$J_H(c;Q) = \{j : \gcd(c + h_i - h_j, M_Q) = 1 \text{ for every } i \neq j\},$$

and the absorption reuse degree

$$\boxed{\mu_H(Q) = \max_{c \in (\mathbb{Z}/M_Q\mathbb{Z})^\times} |J_H(c;Q)|.}$$

The central finite-capacity theorem of this part is then

$$\boxed{D_H(r,s) \leq \mu_H(Q)\,|U(r,s)|.}$$

If

$$\Delta = s - r,$$

then for $r \geq 5$ we also have the structural upper bound

$$\boxed{|U(r,s)| \leq \Delta - 1.}$$

Hence define two finite absorption capacities:

$$\boxed{C_H^{\text{exact}} = \mu_H(Q)|U(r,s)|,}$$

$$\boxed{C_H^{\text{struct}} = \mu_H(Q)(\Delta - 1).}$$

and two net relation potentials:

$$\boxed{e_H^{\text{exact}} = B_H - C_H^{\text{exact}},}$$

$$\boxed{e_H^{\text{struct}} = B_H - C_H^{\text{struct}}.}$$

By the capacity theorem,

$$\boxed{T_H \geq e_H^{\text{exact}} \geq e_H^{\text{struct}}.}$$

Therefore

$$\boxed{e_H^{\text{exact}} > 0 \Rightarrow T_H \geq 1,}$$

while the stronger purely structural certificate is

$$\boxed{e_H^{\text{struct}} > 0 \Rightarrow T_H \geq 1.}$$

Across square horizons define the cumulative net potentials

$$\boxed{\mathcal{E}_H^{\text{exact}}(N) = \sum_{j \leq N} e_{H,j}^{\text{exact}},}$$

$$\boxed{\mathcal{E}_H^{\text{struct}}(N) = \sum_{j \leq N} e_{H,j}^{\text{struct}}.}$$

If

$$T_H^{\leq N} = \sum_{j \leq N} T_{H,j},$$

then summing window by window yields

$$\boxed{T_H^{\leq N} \geq \mathcal{E}_H^{\text{exact}}(N) \geq \mathcal{E}_H^{\text{struct}}(N).}$$

Consequently,

$$\boxed{\limsup_{N \to \infty} \mathcal{E}_H^{\text{exact}}(N) = +\infty \Rightarrow H\text{-relations recur infinitely often.}}$$

Likewise,

$$\boxed{\limsup_{N \to \infty} \mathcal{E}_H^{\text{struct}}(N) = +\infty \Rightarrow H\text{-relations recur infinitely often.}}$$

This gives a rigorous bridge from finite windows to infinite recurrence.

The unit-difference relation

$$H = \{0,2\}$$

is the smallest instance of this general structure. For

$$Q = \{3,5\},$$

we have

$$\boxed{\mu_{\{0,2\}}(Q) = 1.}$$

Hence

$$\boxed{T_{r,s} \geq B_{r,s} - |U(r,s)| \geq B_{r,s} - (\Delta - 1).}$$

Unique square-horizon assignment, seam closure, double escape, single-layer purification, the genuine-death witness

$$v = ru \pm 2,$$

single-cofactor single death, the clean zone, resource half-life, and finite lifetime are all retained and incorporated into this more general supply–absorption framework.

In particular, resource half-life gives

$$\boxed{\log\frac{u}{s} < \frac{1}{2}\log\frac{u}{r}.}$$

This rigorously shows that

every fixed historical atom has only a finite absorption lifetime.

The global closure of infinite recurrence is completed by cumulative capacity. New historical resources are themselves continually born, so this part places **all relation supply** and **all available absorption capacity** on the same cumulative ledger; the reuse degree, half-life, and finite lifetime control the absorption side, while persistent relation supply controls the supply side, and the cumulative net potential yields the final global decision.

This part also retains the relation-level Lucas certificate for unit difference. For a given enormous candidate

$$(M - 1, M + 1),$$

one may use a common center, a common Lucas system, and common group-order constraints to certify the entire relation in one operation; when the relevant 2-power principal chain is sufficiently thick, the complete center-factor information can further be compressed into one sufficiently thick common principal chain.

The finite computation section retains the window-by-window verification of 905 consecutive square horizons, and adds the unified-pattern-module data

$$B_H,\ U,\ \mu_H,\ C_H,\ e_H,\ T_H,\ D_H$$

for each window together with the cumulative net potentials.

The final principle established in this part is

Infinity is not an untouchable whole;
it is the unbounded accumulation of a sequence of finite certificates.

Keywords: infinite recurrence; universal quantifier; persistent non-extinction; square horizon; CRT survival tree; relation alphabet; relation supply; finite absorption; absorption reuse degree; net

relation potential; cumulative certificate; Twin; prime tuples; batch proof; formal proof; relation-level certificate

Part I Infinite Quantifiers Are Not a Single Concept

Chapter 1 From Finite Terminality to Infinite Structure

Part II has already proved:

> Every relation with finitely many endpoints can acquire its final local identities within a finite horizon.

This resolves

> how one concrete relation instance is finitely determined.

But the following propositions plainly do not impose the same requirement:

$$\forall N\ \exists p,$$

$$\exists^{\infty} p,$$

$$\forall k\ \exists a, d,$$

$$\exists^{\infty} \theta\ \exists x,$$

$$\forall n\ P(n).$$

All involve infinitely many parameters, but their quantifier structures differ.

Therefore

> **Infinity is first a quantifier structure,**
> **and only then a scale of computation.**

(1.1)

Making a finite horizon arbitrarily large does not alter this point.

Chapter 2 Seven Principal Global Interfaces

This book divides the common global relations into seven classes.

2.1 Infinite recurrence

Form:

$$\boxed{\exists^{\infty} x: R(x) = 1.} \tag{2.1}$$

Typical relations:

- Twin;
- general fixed difference;
- Sophie Germain;
- fixed prime tuple;
- polynomial prime pullback;
- recurrence of Mersenne-prime parameters.

2.2 Parameterwise universal fibers

Form:

$$\boxed{\forall \theta, \quad \exists x: R_{\theta}(x) = 1.} \tag{2.2}$$

Typical relations:

- Goldbach;
- Lemoine / Lévy;
- other targetwise nonempty reflection fibers.

2.3 Window universality

Form:

$$\boxed{\forall n, \quad \exists x \in I_n: \alpha(x) = 1.} \tag{2.3}$$

Typical relations:

- Legendre;
- the two windows in Oppermann's conjecture.

2.4 Arbitrarily long relations

Form:

$$\boxed{\forall k, \quad \exists \Theta, x: R_{k,\Theta}(x) = 1.} \tag{2.4}$$

Typical relation:

- arbitrarily long prime arithmetic progressions in the Green–Tao theorem.

2.5 Parameter infinitude

Form:

$$\boxed{\exists^{\infty}\theta, \quad \exists x: R_{\theta}(x) = 1.} \tag{2.5}$$

Here it is the successful parameters themselves that are infinite in number.

2.6 Indexwise universal predicates

Form:

$$\boxed{\forall n, \quad P(n).} \tag{2.6}$$

Typical relations:

- Brocard-type window counts;
- Andrica;
- Firoozbakht;
- Grimm-type blockwise relations.

2.7 Identity-block separation

This class requires the existence of a finite block length such that complete factor-identity blocks beginning at different starting points cannot remain identical.

A typical form is

$$\boxed{\exists K\ \forall x \neq y\ \exists 0 \leq j < K: \mathrm{PF}(x+j) \neq \mathrm{PF}(y+j).} \tag{2.7}$$

This kind of relation is entirely different from “occurring infinitely often.”

Figure 5-01 Seven Principal Global Interfaces

Different propositions share the fixed spectrum, but their global criteria differ; each quantifier has its own standard finite quantity.

Quantifier type	Standard finite quantity	Global criterion
Infinite recurrence	C_R(X)	unbounded
Universal fiber by parameter	C(θ)	every parameter >0
Window universality	W(n)	every window ≥1
Arbitrarily long	L(X)	unbounded
Parameter infinitude	P(X)	unbounded
Universal indexed predicate	F(X)	identically 0
Identity-block separation	B_K(x)	one uniform finite K exists

'Infinite' is not one single quantifier; the interface type determines which finite certificate the proof must track.

Figure 5-01. Seven Principal Global Interfaces. Part V divides global relations into seven classes: infinite recurrence, universal fibers by parameter, window universality, arbitrary length, parameter infinitude, universal indexed predicates, and identity-block separation. Their standard finite quantities are respectively $C_R(X), C(\theta), W(n), L(X), P(X), F(X), B_K(x)$; they therefore cannot be replaced by one generic ‘infinity’ criterion.

Chapter 3 The Exact Criterion for Infinite Recurrence

Let

$$R(x) \in \{0,1\}.$$

Define the cumulative hit function

$$\boxed{C_R(X) = \sum_{x \le X} R\,(x).} \tag{3.1}$$

Theorem 3.1 Counting Criterion for Infinite Recurrence

$$\boxed{\exists^{\infty} x: R(x) = 1 \Leftrightarrow C_R(X) \text{ is unbounded.}} \tag{3.2}$$

Proof

If the hit set is infinite, then for every integer M we can choose at least M distinct hit points. Taking X larger than all of them gives

$$C_R(X) \geq M.$$

Hence the cumulative count is unbounded.

Conversely, if the hit set is finite, say with total cardinality M, then

$$C_R(X) \leq M$$

for every X, contradicting unboundedness. This proves the theorem.

This theorem converts

$$\boxed{\exists^{\infty}}$$

completely into

$$\boxed{\text{whether a monotone integer-valued function is unbounded.}}$$

This is the most basic batch interface for infinite recurrence.

Chapter 4 The Exact Criterion for Parameterwise Universal Fibers

Define

$$\boxed{C(\theta) = \sum_{x} R_\theta\,(x).} \tag{4.1}$$

Then:

Theorem 4.1 Nonemptiness Criterion for Universal Fibers

$$\boxed{\forall\theta\ \exists x{:}\ R_\theta(x) = 1 \Leftrightarrow C(\theta) > 0 \quad \forall\theta.} \tag{4.2}$$

Thus the target of a batch proof for universal fibers is typically to construct

$$\boxed{L(\theta) > 0}$$

such that

$$\boxed{C(\theta) \geq L(\theta).} \tag{4.3}$$

This is entirely different from infinite recurrence:

- universality allows no fiber to be missed;
- infinite recurrence permits arbitrarily large gaps, requiring only that there be no final hit.

Chapter 5 Window Universality, Arbitrarily Long Relations, and Parameter Infinitude

5.1 Window universality

Define

$$W(n) = \sum_{x \in I_n} \alpha\,(x).$$

Then

$$\boxed{\forall n\ \exists x \in I_n \cap \mathbb{P} \Leftrightarrow W(n) \geq 1 \quad \forall n.} \tag{5.1}$$

5.2 Arbitrarily long relations

Define the maximal realized length up to horizon X by

$$\boxed{L(X) = \max\{k: \exists \Theta, x \leq X,\ R_{k,\Theta}(x) = 1\}.} \tag{5.2}$$

Then

$$\boxed{\forall k\ \exists \Theta, x: R_{k,\Theta}(x) = 1 \Leftrightarrow L(X) \text{ is unbounded.}} \tag{5.3}$$

5.3 Parameter infinitude

Define

$$\boxed{P(X) = \#\{\theta \leq X: \exists x,\ R_\theta(x) = 1\}.} \tag{5.4}$$

Then

$$\boxed{\exists^\infty \theta\ \exists x: R_\theta(x) = 1 \Leftrightarrow P(X) \text{ is unbounded.}} \tag{5.5}$$

Chapter 6 Indexwise Universality and Identity-Block Separation

6.1 Indexwise universality

Define the failure count

$$\boxed{F(X) = \sum_{n \le X} \mathbf{1}_{\{\neg P(n)\}}.} \tag{6.1}$$

Then

$$\boxed{\forall n,\ P(n) \Leftrightarrow F(X) = 0 \quad \forall X.} \tag{6.2}$$

6.2 Identity-block separation

Define

$$B_K(x) = (\mathrm{PF}(x), \dots, \mathrm{PF}(x + K - 1)).$$

The global target can be written as

$$\boxed{\exists K\ \forall x \neq y : B_K(x) \neq B_K(y).} \tag{6.3}$$

Here the requirement is a **uniform finite separation length**.

Thus it cannot be replaced by ordinary unbounded cumulative hits.

Chapter 7 Quantifiers Determine the Proof Interface

The seven interfaces can be compressed into the following table:

Type	Standard finite quantity	Global criterion
Infinite recurrence	$C_R(X)$	unbounded
Universal fiber	$C(\theta)$	> 0 for every parameter
Window universality	$W(n)$	≥ 1 in every window
Arbitrarily long	$L(X)$	unbounded
Parameter infinitude	$P(X)$	unbounded
Indexwise universality	$F(X)$	identically 0
Identity-block separation	$B_K(x)$	there exists a uniform finite K

Therefore

Different infinite propositions share the fixed spectrum,

but they do not share the same global criterion.

(7.1)

Part II Finite Periodicity and Infinite Persistence

Chapter 8 Exact Periodicity at a Finite Control Level

Fix a finite control depth z.

Define

$$\boxed{W_z = \prod_{\substack{3 \le q \le z \\ q \in \mathbb{P}}} q .} \tag{8.1}$$

For any local state depending only on divisibility residue classes for these primes, the phase repeats exactly modulo W_z.

Therefore

a finite control level has exact periodicity.

(8.2)

On the ordinary odd-integer axis, the corresponding period is

$$2W_z.$$

Chapter 9 Finite Periodicity Is Not Infinite Relation Periodicity

When the control depth increases from

$$z$$

to a new prime layer,

$$W_z$$

changes as well.

Thus the global relation does not repeat indefinitely modulo one fixed finite modulus.

One must distinguish

$$\boxed{S_0\text{:the finite phase cycle after the control layer has been frozen,}}$$

from

$$\boxed{S_\infty\text{:the global quantifier while the control layers continue to grow.}}$$

Hence

finite periodicity organizes the local structure;

the infinite interface governs global persistence.

(9.1)

Figure 5-02 Finite Period S0 and Infinite Persistence S∞

At fixed control depth z, phases are exactly periodic modulo W_z; as new prime-control layers keep entering, the global history is not one fixed finite period.

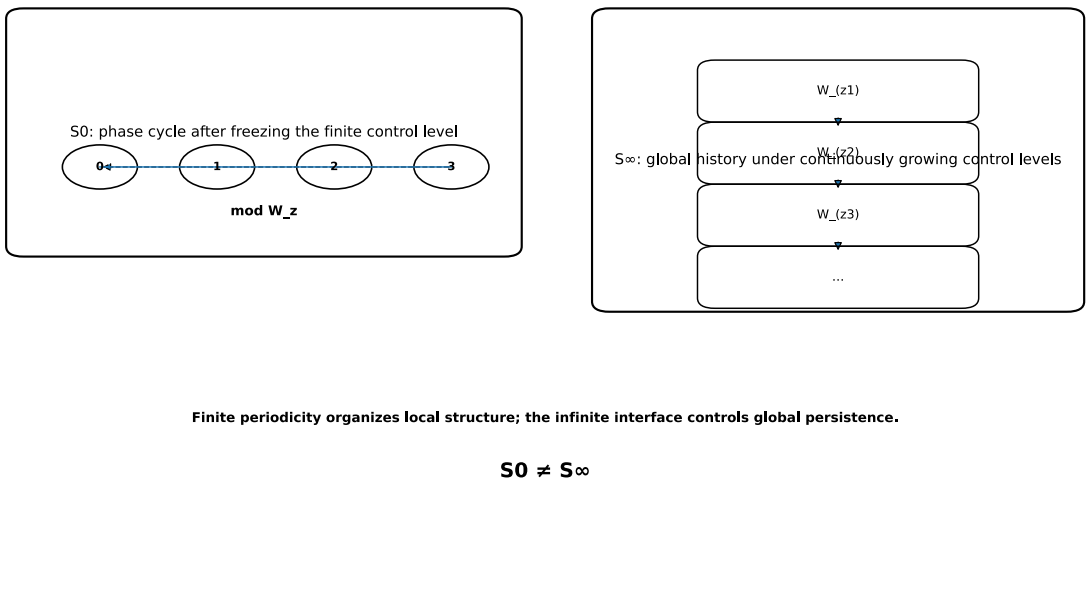


Figure 5-02. Finite Period S0 and Infinite Persistence S∞. At fixed finite control depth z, any local state depending only on those prime-divisibility classes repeats exactly modulo W_z; this is the S_0 level. But as control depth keeps growing, W_z changes as well, so the global relation cannot be

understood as infinite repetition modulo one fixed finite modulus. That is the persistence problem at the S_∞ level.

Chapter 10 Three Kinds of Seams

There are three distinct kinds of boundaries in finite-control geometry.

10.1 Generation seam

A new prime s first becomes capable of producing an exclusively attributable genuine death, as a new least prime factor, at

$$s^2.$$

It answers:

> *When is a new genuine control layer added?*

10.2 Phase seam

A finite control layer returns to the same residue phase modulo

$$W_z.$$

It answers:

> *When do we return to the same phase?*

10.3 Family of square seams

All values

$$r^2$$

form the natural segmentation for consecutive finite identity certification.

They answer:

> *How is the infinite axis partitioned into a sequence of finite verifiable windows?*

Part III Persistent Non-Extinction Relation Sieves

Chapter 11 Why a Fixed Difference Has No Common Terminal Horizon

A fixed-difference relation

$$p, \qquad p + d$$

can be translated arbitrarily far to the right.

Hence there is no finite upper bound X after which the entire fixed-difference fiber can acquire no new members.

Therefore

a fixed-difference relation is not terminal-saturation type,
but persistent non-extinction type.

(11.1)

Each concrete candidate still has a finite terminal horizon;

the entire fiber instead requires

there is no last successful window. (11.2)

Chapter 12 Unit Difference Is the Minimal Persistent Relation

In atomic coordinates define

$$\boxed{\chi_{\mathrm{Twin}}(a) = \hat{\alpha}(a)\hat{\alpha}(a+1).} \tag{12.1}$$

Then

$$\chi_{\mathrm{Twin}}(a) = 1$$

if and only if

$$2a+1, \qquad 2a+3$$

are both prime.

Define the cumulative number of unit-difference hits by

$$\boxed{C_T(X) = \sum_{a \le X} \chi_{\mathrm{Twin}}\,(a) = \sum_{a \le X} \hat{\alpha}\,(a)\hat{\alpha}(a+1).} \tag{12.1a}$$

Then

$C_T(X)$ is unbounded ⇔ unit-difference atomic relations occur infinitely often.

Thus the exact object corresponding to Twin inside the matrix is

$$\boxed{\Gamma^{[1]} = \delta^{-1}(1).} \tag{12.2}$$

and

the proof object is the entire unit-difference relation,
not two unrelated endpoints.

(12.3)

Chapter 13 Matrix-to-Subrelation Inheritance for Unit Difference

If

$$r = \{a, a+1\} \in \Gamma_k,$$

then

$$r \in \Gamma_j \qquad \forall j \geq k.$$

Therefore

once a genuine unit-difference relation is born and certified,
it is never deleted by later atomic history.

(13.1)

This is the persistence of the unit-difference relation.

Chapter 14 Consecutive Prime-Square Horizons

Let

$$\boxed{p < r < s} \tag{14.1}$$

be consecutive odd primes.

Define

$$\boxed{W_{r,s} = [r^2, s^2).} \tag{14.2}$$

Consider

$$r^2 \leq n < n + 2 < s^2.$$

Within this finite window, every genuine unit-difference relation can be certified with zero error.

Chapter 15 Unique Square-Horizon Assignment

Every sufficiently large genuine unit-difference prime pair

$$(q, q+2)$$

belongs to a unique consecutive square horizon.

Indeed,

$$3^2 < 5^2 < 7^2 < \cdots$$

is strictly increasing and tends to infinity.

Distinct square horizons do not overlap.

Hence the infinite unit-difference relation can be decomposed without loss into

$$\boxed{W_1, W_2, W_3, \ldots} \tag{15.1}$$

a sequence of finite worlds.

Chapter 16 Closure of Square Seams

A genuine Twin cannot cross the square boundary of an odd prime.

If

$$q < r^2 < q + 2,$$

then because the spacing between odd integers is 2, the only possibility would involve

$$r^2 - 2, \quad r^2.$$

But

$$r^2$$

is composite.

Therefore

consecutive square horizons form a lossless partition for genuine Twins.

(16.1)

Figure 5-03 Sequence of Square Horizons and Their Seams

Consecutive prime squares partition the infinite axis losslessly into finite horizons; a genuine twin pair does not cross an odd-prime-square boundary.

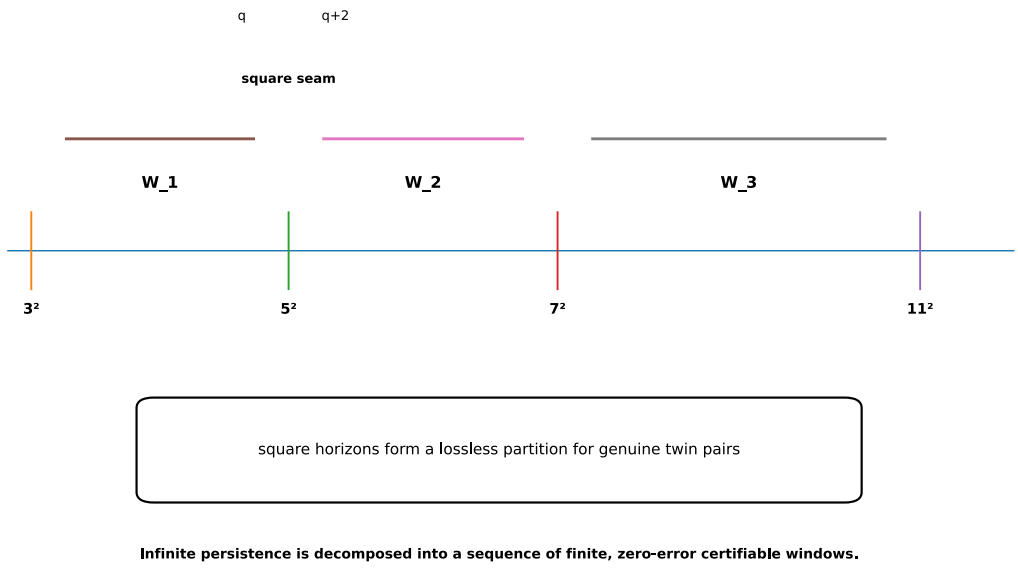


Figure 5-03. Sequence of Square Horizons and Their Seams. Consecutive odd-prime squares partition the infinite axis losslessly into finite horizons $W_{r,s} = [r^2, s^2)$. Every sufficiently large genuine twin pair belongs to exactly one square horizon. If an odd pair were to cross an odd-prime-square boundary, it would have to involve the square itself, which is composite; hence a genuine twin pair does not cross a square seam.

Chapter 17 Old Double-Escape Relations

The old history has already processed every prime factor

$$q \le p.$$

Define

$$\boxed{\mathcal{B}_{r,s} = \left\{(n, n+2): \begin{matrix} r^2 \le n < n+2 < s^2, \\ \gcd(n(n+2), p^{\#}) = 1 \end{matrix}\right\}.} \tag{17.1}$$

Let

$$\boxed{B_{r,s} = |\mathcal{B}_{r,s}|.} \tag{17.2}$$

These candidates simultaneously escape all old historical control layers.

Chapter 18 Single-Layer Purification

For any composite endpoint of an old double-escape relation, the least prime factor can only be

$$r.$$

Hence the only possible terminal identity types are

$$\boxed{(P,P), \qquad (ru,P), \qquad (P,ru).} \tag{18.1}$$

The type

$$(ru, rv)$$

cannot occur, since it would require

$$r \mid 2.$$

Therefore

a square horizon compresses all old double-escape relations into a finite three-state system.

(18.2)

Using the relation-alphabet notation of Part IV, this three-state system is exactly

$$\boxed{\Sigma_{\mathrm{Twin}}^{(r)} = \{AA, \mathcal{M}_r A, A\mathcal{M}_r\}.} \tag{18.3}$$

After terminal certification, every old double-escape candidate belongs to exactly one of these identity words.

Therefore

$$\boxed{B_{r,s} = T_{r,s} + D_{r,s}}$$

is not merely a numerical identity, but also a classification identity in the relation alphabet:

- AA contributes $T_{r,s}$;
- $\mathcal{M}_r A$ and $A\mathcal{M}_r$ together contribute $D_{r,s}$.

Thus the supply–absorption ledger preserves endpoint-level terminal identities from the outset.

Chapter 19 Exact Twin Formula in a Square Horizon

Theorem 19.1

$$\boxed{\mathcal{T}_{r,s} = \{n: r^2 \le n < n+2 < s^2,\ \gcd(n(n+2), r^{\#}) = 1\}.} \tag{19.1}$$

Proof

If

$$\gcd(n(n+2), r^{\#}) = 1,$$

then neither endpoint has a prime factor at most r.

If one of the endpoints were composite, then because it is less than s^2, it would have a prime factor less than s.

Since r and s are consecutive primes, that prime factor would have to be at most r, a contradiction.

The converse is immediate. This proves the theorem.

Therefore

within a single square horizon, Twin is a completely finite, exact, zero-error object.

(19.2)

Chapter 20 Relation Death Leaves a Three-Atom Witness

If an old double-escape relation is not a Twin, then either

$$n = ru, \qquad n + 2 = v,$$

or

$$n = v, \qquad n + 2 = ru,$$

where u, v are primes.

Hence

$$\boxed{v = ru \pm 2.} \tag{20.1}$$

Thus

relation failure is not deletion, but a transformation of relation identity.

(20.2)

A dead relation leaves the complete finite witness

$$(r, u, v).$$

Chapter 21 One Cofactor, One Death

Fix

$$r, u > 3.$$

Since

$$ru \equiv \pm 1 \pmod 3,$$

at least one of

$$ru - 2, \qquad ru + 2$$

is divisible by 3.

Thus at most one of them can be a prime greater than 3.

Therefore

for a fixed current control atom r and cofactor u, the same u can absorb at most one unit-differe	(21.1)

Chapter 22 The Clean Zone and the Absorption Zone

The square horizon decomposes as

$$(r^2, rs) \quad \sqcup \quad [rs, s^2).$$

Suppose

$$r^2 < n < n + 2 < rs$$

and the pair escapes the old history. If a composite endpoint were absorbed by r, it would have to be of the form

$$ru$$

with

$$u \geq s.$$

Then

$$ru \geq rs,$$

a contradiction.

Hence

every old double-escape relation in the clean zone is directly a genuine Twin.

(22.1)

Internal absorption can begin only in

$$[rs, s^2).$$

Chapter 23 The Finite Absorption Resource Pool

If

$$ru \in [rs, s^2),$$

then

$$s \le u < \frac{s^2}{r}.$$

Define

$$\boxed{U(r,s) = \left\{ u \in \mathbb{P} : s \le u < \frac{s^2}{r} \right\}.} \tag{23.1}$$

If one uses a closed horizon containing the lower square boundary, one may instead define

$$\boxed{U_{r,s}^{\text{full}} = \left\{ u \in \mathbb{P} : r \le u < \frac{s^2}{r} \right\}.} \tag{23.2}$$

The two conventions differ only by a finite boundary layer.

Therefore

all absorption capacity in a square horizon is already a finite set when the horizon opens.

(23.3)

Chapter 24 Death Matching and Finite Forcing

Let

$$D_{r,s} = \text{number of dead relations},$$

$$T_{r,s} = \text{number of genuine Twins}.$$

Then

$$\boxed{B_{r,s} = T_{r,s} + D_{r,s}.} \tag{24.1}$$

By the one-cofactor one-death property, each dead relation can be mapped to the cofactor u responsible for its absorption.

Hence

$$\boxed{\mathcal{D}_{r,s} \hookrightarrow U_{r,s}^{\text{full}}.} \tag{24.2}$$

Therefore

$$\boxed{D_{r,s} \leq |U_{r,s}^{\text{full}}|.} \tag{24.3}$$

Consequently

$$\boxed{T_{r,s} \geq B_{r,s} - |U_{r,s}^{\text{full}}|.} \tag{24.4}$$

In particular,

$$\boxed{B_{r,s} > |U_{r,s}^{\text{full}}| \Rightarrow T_{r,s} \geq 1.} \tag{24.5}$$

This is the finite-window forcing condition for unit difference.

Chapter 25 Local Moving Intervals

A square horizon is not the smallest usable scale.

Take any

$$I = [X, Y] \subset [r^2, s^2).$$

Let

$$B_r(I)$$

be the number of old double-escape relations in I, and define

$$\boxed{U_r(I) = \{u \in \mathbb{P} : ru \in I \cup (I + 2)\}.} \tag{25.1}$$

Then likewise

$$\boxed{T_r(I) \geq B_r(I) - |U_r(I)|.} \tag{25.2}$$

Hence

$$\boxed{B_r(I) > |U_r(I)| \Rightarrow I \text{ contains at least one Twin pair.}} \tag{25.3}$$

Chapter 26 Scale Separation Between Relation Space and Resource Space

Let

$$\Delta = s - r.$$

The length of the square horizon is

$$s^2 - r^2 = \Delta(r + s) \sim 2r\Delta.$$

The clean-zone length is

$$\boxed{rs - r^2 = r\Delta.} \tag{26.1}$$

The absorption-zone length is

$$\boxed{s^2 - rs = s\Delta.} \tag{26.2}$$

The cofactor resource-window length is

$$\boxed{\frac{s^2}{r} - s = \frac{s}{r}\Delta.} \tag{26.3}$$

By Bertrand's postulate,

$$1 < \frac{s}{r} < 2,$$

so

$$\boxed{\Delta < \frac{s^2}{r} - s < 2\Delta.} \tag{26.4}$$

Thus

$$\boxed{\begin{array}{c}\text{scale of relation-position space} \sim r\Delta, \\ \text{scale of the absorption-resource window} \sim \Delta.\end{array}} \tag{26.5}$$

Resource space is lower by one factor of r than relation-position space.

Chapter 27 A Linear Upper Bound for Absorption Resources

Because u and s are odd primes, write

$$u = s + 2m.$$

From

$$u < \frac{s^2}{r}$$

we obtain

$$2m < \frac{s}{r}(s - r).$$

Hence

$$\boxed{|U(r,s)| \leq \left\lceil \frac{s(s-r)}{2r} \right\rceil}. \tag{27.1}$$

This is an immediate finite upper bound.

A stronger purely structural bound will be established in the general-pattern section.

Chapter 28 Resource Half-Life

Fix a historical atom u.

If it is eligible for absorption in the horizon (r, s), then

$$u < \frac{s^2}{r}.$$

Equivalently,

$$s > \sqrt{ru}.$$

Define

$$\boxed{d(r,u) = \log\frac{u}{r}.} \tag{28.1}$$

Then

$$\boxed{\log\frac{u}{s} < \frac{1}{2}\log\frac{u}{r}.} \tag{28.2}$$

Thus each time a new absorption eligibility is obtained, the remaining logarithmic distance is reduced by at least one half.

Theorem 28.1 Resource Half-Life Theorem

If the same u is eligible for absorption in several successive preceding square horizons, then its logarithmic distance to its own position decreases at each stage by at least a factor of one half.

Therefore

the same historical atom cannot continue to play an absorption role in infinitely many preceding square horizons.

(28.3)

Chapter 29 Resource Lifetime

Let

$$u^{-}$$

be the prime immediately preceding u.

Define

$$\boxed{\mathfrak{Q}_r(u) = \log_2 \frac{\log(u/r)}{\log(u/u^{-})}.} \tag{29.1}$$

Each time a new absorption eligibility is obtained,

$$\mathfrak{Q}_s(u) < \mathfrak{Q}_r(u) - 1.$$

If

$$\rho(u)$$

denotes the number of times u can enter an absorption resource pool before the frontier reaches u itself, then

$$\boxed{\rho(u) \le 1 + \left\lfloor \log_2 \frac{\log(u/3)}{\log(u/u^{-})} \right\rfloor.} \tag{29.2}$$

Hence

$$\boxed{\rho(u) < \infty.} \tag{29.3}$$

and crudely

$$\boxed{\rho(u) = O(\log u).} \tag{29.4}$$

Therefore

every historical atom has only a finite absorption lifetime.

(29.5)

Chapter 30 Finite Lifetime and Global Capacity Are Two Distinct Levels

The resource-lifetime theorem controls

how many times a fixed resource u can be reused.

But the global infinite problem also contains

new resources continually entering history as the frontier advances.

Thus the global decision must accumulate the available capacity of all resources.

Suppose that in the j-th square horizon,

$$B_j = T_j + D_j.$$

Summing across horizons gives

$$\boxed{\sum_{j=m}^{n} T_j = \sum_{j=m}^{n} B_j - \sum_{j=m}^{n} D_j\,.} \tag{30.1}$$

If a uniform capacity upper bound is established for the total number of deaths,

$$\sum D_j \le \mathcal{C}_{m,n},$$

then

$$\boxed{\sum_{j=m}^{n} T_j \ge \sum_{j=m}^{n} B_j - \mathcal{C}_{m,n}.} \tag{30.2}$$

This is the genuine global ledger required to pass from finite lifetime to an infinite quantifier.

Part IV Supply–Absorption Theory for General Fixed Patterns

Chapter 31 From Twin to an Arbitrary Fixed Offset Pattern

Let

$$\boxed{H = \{h_1, \dots, h_k\} \subset \mathbb{Z}.} \tag{31.1}$$

By a global translation we may assume

$$\min H = 0.$$

If the target is a relation among sufficiently large odd primes, then all offsets must have the same parity; otherwise modulus 2 already creates a local obstruction.

In what follows, assume H has been normalized in this way.

Chapter 32 Square Horizons for Patterns

Let

$$p < r < s$$

be consecutive odd primes.

Define

$$\boxed{I_H(r,s) = \{n \in 2\mathbb{Z}+1 : r^2 < n + h < s^2 \quad \forall h \in H\}.} \tag{32.1}$$

This means:

> *the entire pattern $n + H$ lies inside the same open square horizon.*

The open-interval convention separates the finitely many special terms on the lower square boundary from the interior resources.

Chapter 33 Relation Supply B_H

Define the old-history supply set

$$\boxed{\mathcal{B}_H(r,s) = \{n \in I_H(r,s) : P^-(n+h) \geq r \quad \forall h \in H\}.} \tag{33.1}$$

and define

$$\boxed{B_H(r,s) = |\mathcal{B}_H(r,s)|.} \tag{33.2}$$

It counts

> *complete relation patterns that simultaneously escape every old control prime $q < r$.*

Here "supply" means relation supply, not the supply of individual integers.

The CRT survival tree of Part IV has already expressed "escaping the old control primes" as a finite union of surviving phase classes. Let

$$S_{<r} = \{q \in \mathbb{P} : q < r\},$$

and denote the corresponding jointly admissible phase set by

$$A_{<r}(H).$$

Then

$$\boxed{\mathcal{B}_H(r,s) = \text{the actual integer realizations, inside the square horizon, of the jointly surviving CR}} \tag{33.3}$$

Thus one must distinguish

$$|A_{<r}(H)|$$

from

$$B_H(r,s).$$

The former is a finite modular phase capacity; the latter is the complete relation supply actually realized by those phases in the current integer window.

Therefore

> **the CRT survival tree gives admissible branches;**
>
> **B_H gives the relation realizations of those branches in the actual horizon.**

(33.4)

Chapter 34 Genuine Relations T_H and Deaths D_H

Define

$$\boxed{\mathcal{T}_H(r,s) = \{n \in I_H(r,s) : n + h \in \mathbb{P} \quad \forall h \in H\}.} \tag{34.1}$$

Let

$$\boxed{T_H(r,s) = |\mathcal{T}_H(r,s)|.} \tag{34.2}$$

Define the death set

$$\boxed{\mathcal{D}_H(r,s) = \mathcal{B}_H(r,s) \backslash \mathcal{T}_H(r,s),} \tag{34.3}$$

and

$$\boxed{D_H(r,s) = B_H(r,s) - T_H(r,s).} \tag{34.4}$$

Thus

$$\boxed{B_H = T_H + D_H.} \tag{34.5}$$

Recording terminal identities in the relation alphabet, this may be written as

$$\boxed{\mathcal{B}_H(r,s) = \mathcal{B}_H^{\text{all A}}(r,s) \sqcup \mathcal{B}_H^{\text{death}}(r,s).} \tag{34.6}$$

where

$$|\mathcal{B}_H^{\text{all A}}| = T_H, \qquad |\mathcal{B}_H^{\text{death}}| = D_H.$$

For a general pattern, a death identity word may contain one or more $\mathcal{M}_r$-endpoints. The capacity problem of this part is precisely: how many finite resources can these death identity words consume at most?

Chapter 35 The General Form of Single-Layer Purification

Assume

$$r \geq 5.$$

Take

$$n \in \mathcal{B}_H(r, s)$$

but

$$n \notin \mathcal{T}_H(r, s).$$

Then at least one endpoint

$$n + h_j$$

is composite.

Because it has escaped all prime factors

$$q < r,$$

we have

$$P^-(n + h_j) \geq r.$$

Also, since

$$n + h_j < s^2,$$

if its least prime factor were greater than r, it would be at least s, and the composite number would therefore be at least

$$s^2,$$

a contradiction.

Hence

$$\boxed{P^-(n + h_j) = r.} \tag{35.1}$$

Thus

$$n + h_j = ru.$$

Since

$$n + h_j > r^2,$$

we have

$$u > r.$$

By consecutiveness of the primes,

$$u \geq s.$$

If u were composite, then its least prime factor would be at least r, so

$$u \geq r^2,$$

and therefore

$$n + h_j = ru \geq r^3.$$

By Bertrand's postulate,

$$s < 2r,$$

and hence for $r \geq 5$,

$$r^3 > 4r^2 > s^2,$$

a contradiction.

Therefore

$$\boxed{u \in \mathbb{P}.} \tag{35.2}$$

and

$$\boxed{s \leq u < \frac{s^2}{r}.} \tag{35.3}$$

Thus every interior death must consume one finite prime resource.

Chapter 36 The Unified Absorption Resource Pool

Define

$$\boxed{U(r,s) = \left\{u \in \mathbb{P} : s \le u < \frac{s^2}{r}\right\}.} \tag{36.1}$$

This resource pool depends only on the square horizon, not on the particular pattern H.

The pattern determines instead:

> *through how many endpoint roles can a given resource u absorb relations at most?*

This is precisely the origin of the absorption reuse degree.

Chapter 37 The Wheel and the Absorption Reuse Degree

Choose a finite set of primes

$$\boxed{Q \subset \{q \in \mathbb{P} : q < r\}.} \tag{37.1}$$

Define

$$\boxed{M_Q = \prod_{q \in Q} q\,.} \tag{37.2}$$

Suppose a dead relation is absorbed by the current prime r through endpoint j:

$$n + h_j = ru.$$

Then for another endpoint $i \neq j$,

$$n + h_i = ru + h_i - h_j.$$

Since the entire relation belongs to the old-history supply set, for every

$$q \in Q$$

we must have

$$q \nmid ru + h_i - h_j.$$

Let

$$c \equiv ru \pmod{M_Q}.$$

Because $q < r \leq u$ and neither r nor u is one of the primes in Q,

$$\gcd(c, M_Q) = 1.$$

Define

$$\boxed{J_H(c; Q) = \{j : \gcd(c + h_i - h_j, M_Q) = 1 \quad \forall i \neq j\}.} \tag{37.3}$$

and then

$$\boxed{\mu_H(Q) = \max_{c \in (\mathbb{Z}/M_Q\mathbb{Z})^\times} |J_H(c; Q)|.} \tag{37.4}$$

We call this the **absorption reuse degree** of the pattern H under the wheel Q.

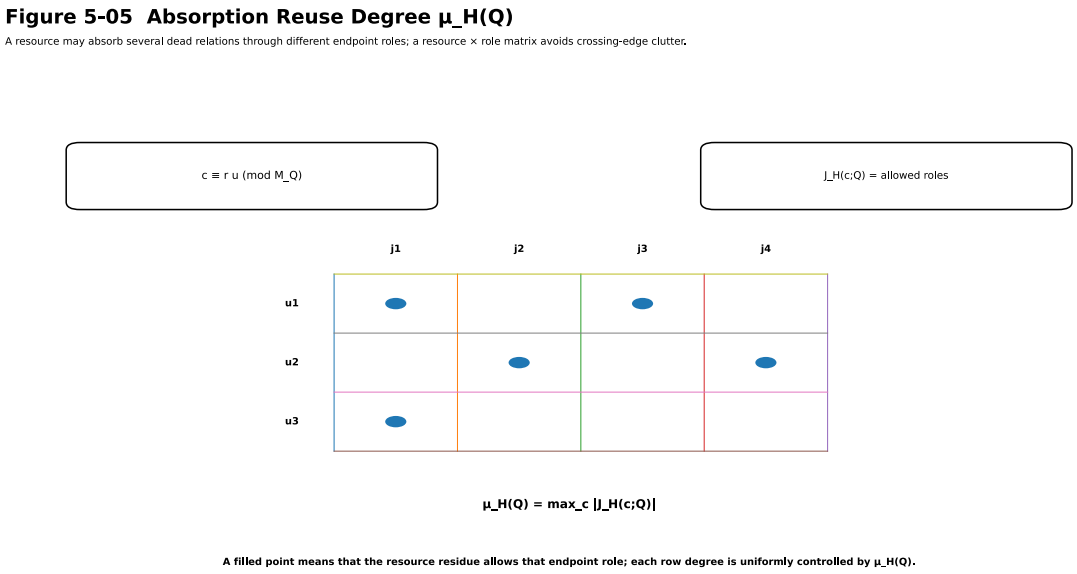


Figure 5-05. Absorption Reuse Degree μ_H(Q). For a wheel Q, a resource u determines the residue $c \equiv ru \pmod{M_Q}$. The endpoint roles j through which that resource may absorb dead relations are determined by $J_H(c;Q)$. Hence the pattern reuse degree $\mu_H(Q) = \max_c |J_H(c;Q)|$ controls how many relation deaths one resource can support, yielding $D_H(r,s) \le \mu_H(Q)|U(r,s)|$.

Chapter 38 The Absorption-Capacity Theorem

Theorem 38.1 Upper Bound on Pattern Death Capacity

For $r \geq 5$,

$$\boxed{D_H(r,s) \leq \mu_H(Q)\,|U(r,s)|.} \tag{38.1}$$

Proof

For each dead relation n, choose the lowest-indexed dead endpoint j such that

$$n + h_j = ru$$

with

$$u \in U(r,s).$$

Map

$$n \mapsto (u,j).$$

For fixed (u,j),

$$n = ru - h_j$$

is unique, so this map is injective into the set of all admissible resource-role pairs.

For fixed u, let

$$c \equiv ru \pmod{M_Q}.$$

Since the relation survives all old control layers, the endpoint role j must belong to

$$J_H(c;Q).$$

Thus a fixed u admits at most

$$|J_H(c;Q)| \leq \mu_H(Q)$$

absorption roles.

Summing over all

$$u \in U(r,s)$$

gives

$$D_H(r,s) \leq \mu_H(Q)|U(r,s)|.$$

This proves the theorem.

This theorem compresses a potentially complicated set of dead relations into

$$\boxed{\text{number of finite resources} \times \text{maximum reuse degree.}}$$

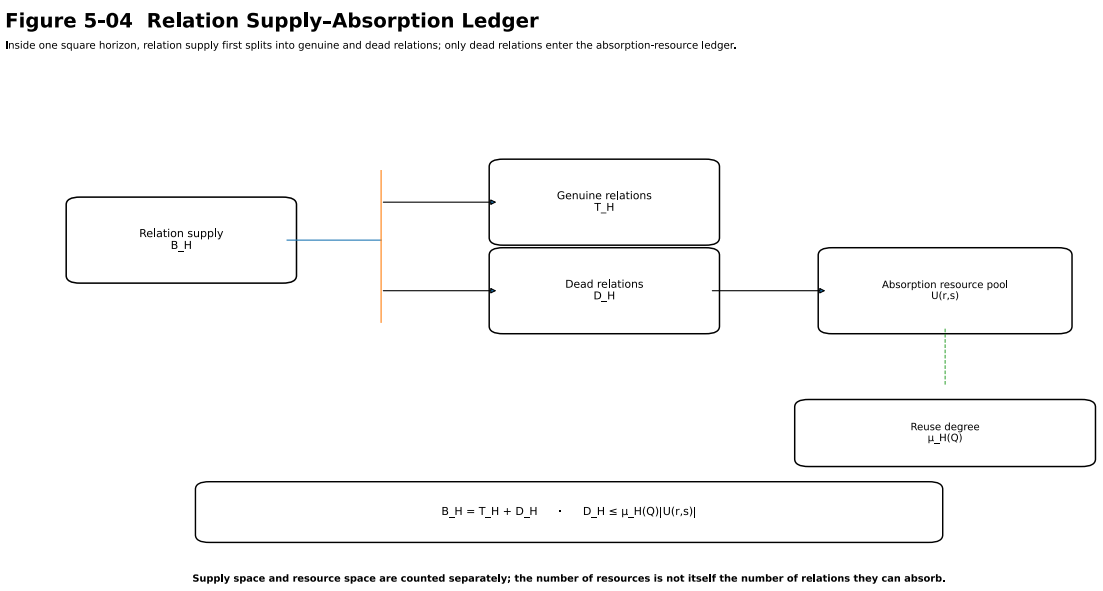


Figure 5-04. Relation Supply–Absorption Ledger. Within one square horizon, admissible relation supply B_H splits into genuine relations T_H and deaths D_H. Dead relations must be absorbed by the finite resource pool $U(r,s)$, while the template reuse degree $\mu_H(Q)$ under wheel Q controls total absorption capacity.

Chapter 39 A Purely Structural Upper Bound for the Resource Pool

Let

$$\Delta = s - r.$$

By Bertrand's postulate,

$$s < 2r.$$

Therefore

$$\frac{s^2}{r} = s + \frac{s\Delta}{r} < s + 2\Delta.$$

Hence primes in

$$U(r, s)$$

can occur only at the odd positions

$$s,\ s+2, \dots, s+2(\Delta-1),$$

at most Δ such positions.

If

$$\Delta \geq 4,$$

then among the first three odd integers

$$s,\ s+2,\ s+4$$

one is divisible by 3; because $s > 3$, that position cannot be prime.

If

$$\Delta = 2$$

and $r \geq 5$, then among the three consecutive odd integers

$$r,\ s = r+2,\ s+2 = r+4$$

one is divisible by 3. Since r and s are primes greater than 3,

$$s + 2$$

must be composite.

Therefore

$$\boxed{|U(r,s)| \leq \Delta - 1 \qquad (r \geq 5).} \tag{39.1}$$

The smallest square horizon $r = 3, s = 5$ is a finite boundary case and can be computed directly.

Chapter 40 Exact Capacity and Structural Capacity

Define

$$\boxed{C_H^{\text{exact}}(r,s;Q) = \mu_H(Q)|U(r,s)|.} \tag{40.1}$$

It uses the actual number of resources from the fixed spectrum.

Define

$$\boxed{C_H^{\text{struct}}(r,s;Q) = \mu_H(Q)(\Delta - 1).} \tag{40.2}$$

It uses only the gap between consecutive primes and the pattern reuse degree.

By Theorem 39.1,

$$\boxed{C_H^{\text{exact}} \le C_H^{\text{struct}}.} \tag{40.3}$$

Figure 5-06 Exact Capacity and Structural Capacity

Exact capacity uses the actual resource count |U(r,s)|; structural capacity uses only the adjacent-prime gap Δ and reuse degree μ_H(Q), giving a provable upper bound.

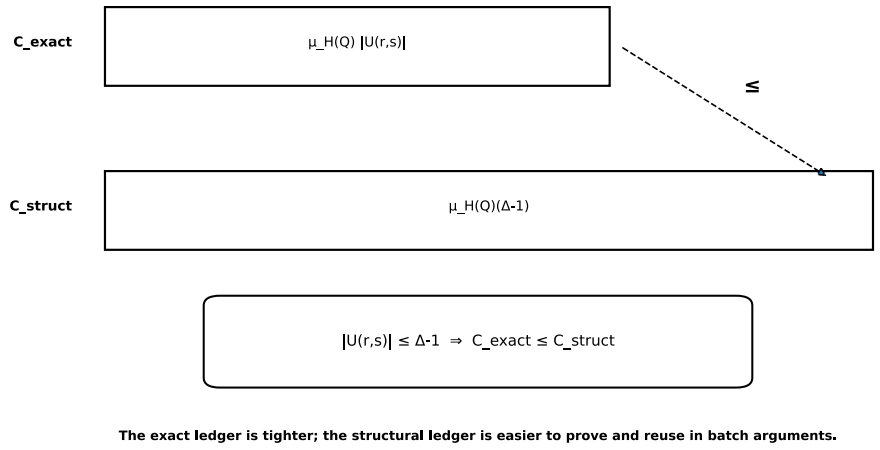


Figure 5-06. Exact Capacity and Structural Capacity. The exact capacity $C_H^{\text{exact}}(r,s;Q) = \mu_H(Q)|U(r,s)|$ uses the actual number of resources in the fixed spectrum. The structural capacity $C_H^{\text{struct}}(r,s;Q) = \mu_H(Q)(\Delta - 1)$ uses only the adjacent-prime gap $\Delta = s - r$ and the reuse degree. Since $|U(r,s)| \le \Delta - 1$, one obtains $C_H^{\text{exact}} \le C_H^{\text{struct}}$.

Chapter 41 Net Relation Potential

Define

$$\boxed{e_H^{\text{exact}} = B_H - C_H^{\text{exact}},} \tag{41.1}$$

$$\boxed{e_H^{\text{struct}} = B_H - C_H^{\text{struct}}.} \tag{41.2}$$

From

$$B_H = T_H + D_H$$

and

$$D_H \le C_H^{\text{exact}} \le C_H^{\text{struct}},$$

we obtain:

Theorem 41.1 Lower Bound from the Single-Horizon Net Relation Potential

$$\boxed{T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}.} \tag{41.3}$$

In particular,

$$\boxed{e_H^{\text{exact}} > 0 \Rightarrow T_H \ge 1,} \tag{41.4}$$

and

$$\boxed{e_H^{\text{struct}} > 0 \Rightarrow T_H \ge 1.} \tag{41.5}$$

Because T_H is an integer, any strictly positive net potential leaves at least one genuine all-atomic relation.

Figure 5-07 Single-Horizon Net Relation Potential

Net potential is relation supply minus a provable maximal absorption capacity; positive, zero, and negative states have different single-horizon implications.

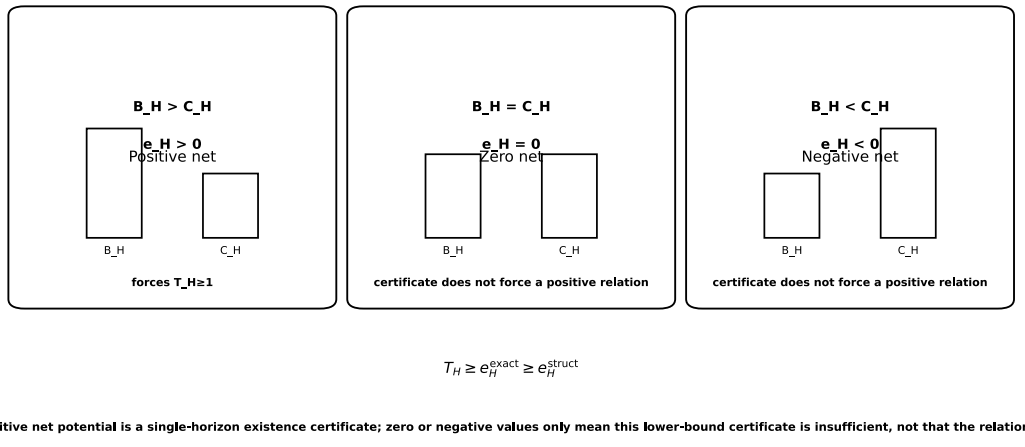


Figure 5-07. Single-Horizon Net Relation Potential. The single-horizon net relation potentials are $e_H^{\text{exact}} = B_H - C_H^{\text{exact}}$ and $e_H^{\text{struct}} = B_H - C_H^{\text{struct}}$, with $T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}$. Thus a strictly positive net potential gives a finite existence certificate $T_H \ge 1$; zero or negative values only say that this lower-bound certificate is insufficient and do not imply absence of the relation.

Chapter 42 Two Types of Finite Relation Certificates

We therefore obtain two standard finite certificates.

42.1 Fixed-spectrum resource certificate

If

$$\boxed{B_H > \mu_H(Q)|U|,} \tag{42.1}$$

then

$$T_H \geq 1.$$

This certificate uses the actual resource count in the current fixed spectrum.

42.2 Structural certificate

If

$$\boxed{B_H > \mu_H(Q)(\Delta - 1),} \tag{42.2}$$

then

$$T_H \geq 1.$$

It does not require enumeration of the actual primes in U; it uses only

- the relation supply B_H;
- the pattern reuse degree $\mu_H(Q)$;
- the consecutive-prime gap Δ.

Thus

the structural certificate is coarser than the fixed-spectrum resource certificate,
but is closer to a unified form suitable for batch transport.

(42.3)

The two certificates have the same logical shape:

$$\boxed{\text{finite input data} + \text{a proved capacity upper bound} \Rightarrow \text{the current relation fiber is nonempty.}} \tag{42.4}$$

Thus, once a certificate has been established, it can be invoked directly by a higher-level formal proof.

Chapter 43 Several Basic Values of the Absorption Reuse Degree

Take

$$Q = \{3,5\}.$$

For several common patterns one computes directly:

$$\boxed{\mu_{\{0,2\}} = 1,}$$

$$\boxed{\mu_{\{0,4\}} = 1,}$$

$$\boxed{\mu_{\{0,6\}} = 2,}$$

$$\boxed{\mu_{\{0,2,6\}} = 2,}$$

$$\boxed{\mu_{\{0,2,6,8\}} = 1.}$$

These values show:

> More endpoints in a pattern do not imply that one absorption resource may be reused arbitrarily many times.

The internal difference structure of the pattern restricts the admissible absorption roles.

Chapter 44 Twin Is the Minimal Model with $\mu = 1$

Take

$$H = \{0,2\}.$$

For

$$Q = \{3,5\},$$

we have

$$\mu_H(Q) = 1.$$

Therefore

$$\boxed{D_{r,s} \leq |U(r,s)|.} \tag{44.1}$$

and

$$\boxed{T_{r,s} \geq B_{r,s} - |U(r,s)|.} \tag{44.2}$$

Using

$$|U(r,s)| \leq \Delta - 1,$$

we obtain

$$\boxed{T_{r,s} \geq B_{r,s} - (\Delta - 1).} \tag{44.3}$$

Thus “one cofactor, one death” is precisely the statement

$$\boxed{\mu_{\{0,2\}} = 1}$$

within the general pattern-capacity theorem.

Part V From a Single Horizon to Infinite Recurrence

Chapter 45 The Cumulative Ledger Across Horizons

Index the square horizons by

$$W_1, W_2, \ldots$$

For a fixed pattern H, write

$$B_{H,j}, \quad T_{H,j}, \quad D_{H,j},$$

$$C_{H,j}^{\text{exact}}, \quad C_{H,j}^{\text{struct}},$$

$$e_{H,j}^{\text{exact}}, \quad e_{H,j}^{\text{struct}}.$$

For each window,

$$T_{H,j} \geq e_{H,j}^{\text{exact}} \geq e_{H,j}^{\text{struct}}.$$

Define the cumulative number of genuine relations by

$$\boxed{\mathcal{T}_H(N) = \sum_{j \leq N} T_{H,j}\,.} \tag{45.1}$$

Define the cumulative net potentials by

$$\boxed{\mathcal{E}_H^{\text{exact}}(N) = \sum_{j \leq N} e_{H,j}^{\text{exact}},} \tag{45.2}$$

$$\boxed{\mathcal{E}_H^{\text{struct}}(N) = \sum_{j \leq N} e_{H,j}^{\text{struct}}.} \tag{45.3}$$

Summing term by term gives

$$\boxed{\mathcal{T}_H(N) \geq \mathcal{E}_H^{\text{exact}}(N) \geq \mathcal{E}_H^{\text{struct}}(N).} \tag{45.4}$$

Chapter 46 Infinite Criteria from Cumulative Net Potential

Theorem 46.1 Infinite Criterion from Exact Resources

If

$$\boxed{\limsup_{N\to\infty}\mathcal{E}_H^{\text{exact}}(N) = +\infty,} \tag{46.1}$$

then all-atomic relations of pattern H recur infinitely often.

Proof

Since

$$\mathcal{T}_H(N) \geq \mathcal{E}_H^{\text{exact}}(N),$$

and the limsup of the right-hand side is $+\infty$,

$$\mathcal{T}_H(N)$$

is unbounded.

By the counting criterion for infinite recurrence, genuine relations occur infinitely often. This proves the theorem.

Theorem 46.2 Structural Infinite Criterion

If

$$\boxed{\limsup_{N\to\infty}\mathcal{E}_H^{\text{struct}}(N) = +\infty,} \tag{46.2}$$

then all-atomic relations of pattern H recur infinitely often.

Proof

Since

$$\mathcal{E}_H^{\text{exact}}(N) \geq \mathcal{E}_H^{\text{struct}}(N),$$

this follows immediately from Theorem 46.1. This proves the theorem.

This is the central bridge in this part from finite relation sieves to infinite quantifiers.

It compresses infinitude into

whether the cumulative net potential of finite relations is unbounded.

(46.3)

Figure 5-08 Cumulative Net Potential and Infinite Recurrence

Single-horizon net potential is e_j=B_j-C_j; after accumulation across horizons, the cumulative certificate determines entry into infinite recurrence.

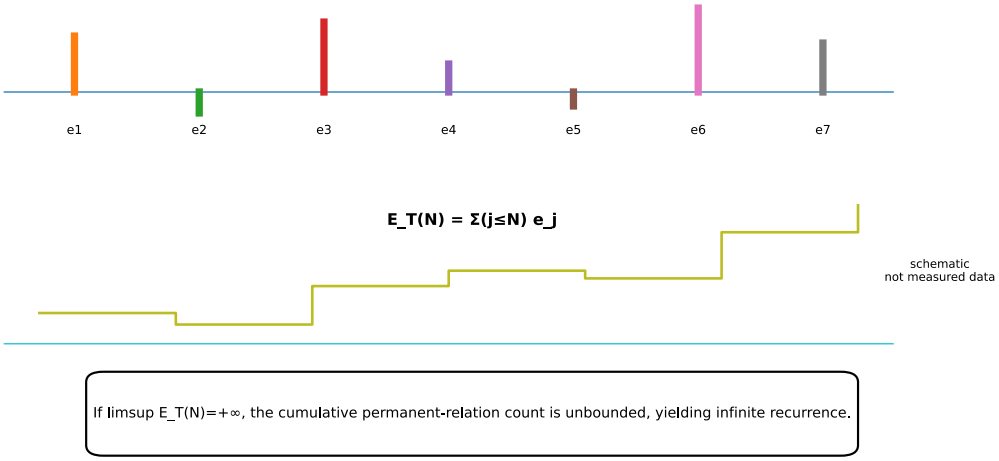


Figure 5-08. Cumulative Net Potential and Infinite Recurrence. Each horizon produces a single-layer net potential $e_j = B_j - C_j$, and the horizons combine into the cumulative ledger $\mathcal{E}_T(N) = \sum_{j \le N} e_j$. When cumulative net potential is unbounded, the cumulative number of permanent relations is unbounded, taking finite certificates into the infinite-recurrence quantifier.

Chapter 47 Stronger Window-by-Window Certificates

If, from some horizon onward,

$$e_{H,j}^{\mathrm{struct}} > 0$$

holds for every j, then every subsequent square horizon contains at least one genuine relation.

This plainly implies infinite recurrence.

More weakly, it is enough that

$$e_{H,j}^{\mathrm{exact}} > 0$$

hold for infinitely many j.

The cumulative-net-potential criterion is more general, because it allows some windows to have negative net potential and requires only that the long-term cumulative value not be suppressed.

Chapter 48 Equivalent Form of “There Is No Last Successful Horizon”

For a fixed pattern H, the following are equivalent:

1. H-relations recur infinitely often;
2. $\mathcal{T}_H(N)$ is unbounded;
3. for every K, there exists $j > K$ such that

$$T_{H,j} > 0.$$

Therefore

$$\boxed{\text{infinite recurrence} = \text{there is no last square horizon that produces a new genuine relation.}} \tag{48.1}$$

This statement now has a rigorous counting meaning, rather than being merely intuitive.

Chapter 49 The Exact Role of Single-Resource Lifetime in the Cumulative Architecture

Resource half-life and finite lifetime remain essential.

They show that

no existing resource can bear infinite absorption responsibility forever.

But as the frontier advances, new resources continually enter history.

Thus cumulative capacity must simultaneously record

resource birth + single-resource reuse count + pattern reuse degree + horizon resource pool.

Hence

single-resource lifetime controls longitudinal reuse;

cumulative net potential controls the total transverse capacity of newly arriving resources.

(49.1)

Together they constitute the complete resource ledger of a persistent non-extinction relation sieve.

Chapter 50 Local Admissibility, Finite Net Potential, and Infinite Recurrence Are Three Distinct Conclusions

For a fixed pattern H, three levels must be kept separate.

Level 1 Local admissibility

$$\nu_H(q) < q \quad \forall q.$$

This says only that no single local prime layer completely blocks the entire pattern.

Level 2 Finite-horizon certificate

$$e_H^{\mathrm{exact}} > 0$$

or

$$e_H^{\mathrm{struct}} > 0.$$

This says that a particular square horizon contains at least one genuine relation.

Level 3 Infinite recurrence

$$\limsup \mathcal{E}_H(N) = +\infty.$$

This says that the total number of genuine relations is unbounded.

Therefore

$$\boxed{\text{local admissibility} \not\equiv \text{finite hit} \not\equiv \text{infinite recurrence.}} \tag{50.1}$$

The role of the unified theory is to construct exact interfaces between these three levels.

Part VI The Place of Twin in the Exact Infinite Architecture

Chapter 51 The Complete Finite State for Twin

For each square horizon, the Twin data can be compressed into

$$\boxed{(r, s, \Delta, B, U, \mu, C_{\text{exact}}, C_{\text{struct}}, e_{\text{exact}}, e_{\text{struct}}, T, D).} \tag{51.1}$$

where

- r, s: consecutive control primes;
- $\Delta = s - r$: the prime gap;
- B: supply of old double-escape relations;
- U: number of internal absorption resources;
- $\mu = 1$: absorption reuse degree of the unit-difference pattern;
- $C_{\text{exact}} = |U|$;
- $C_{\text{struct}} = \Delta - 1$;
- $e_{\text{exact}} = B - |U|$;
- $e_{\text{struct}} = B - (\Delta - 1)$;
- T: number of genuine Twins;
- $D = B - T$: number of genuine deaths.

This is the complete window-by-window record for the infinite interface of unit difference.

Chapter 52 How the Global Twin Proposition Is Expressed

Define

$$\boxed{\mathcal{T}(N) = \sum_{j \le N} T_j \,.} \tag{52.1}$$

Then the infinite-recurrence form of Twin is

$$\boxed{\mathcal{T}(N) \text{ is unbounded.}} \tag{52.2}$$

Equivalently,

$$\boxed{\forall K,\ \exists j > K : T_j > 0.} \tag{52.3}$$

The supply–absorption framework gives the sufficient criterion

$$\boxed{\limsup_{N\to\infty} \sum_{j \le N} \left(B_j - |U_j|\right) = +\infty} \tag{52.4}$$

or the stronger purely structural form

$$\boxed{\limsup_{N\to\infty} \sum_{j \le N} \left(B_j - (\Delta_j - 1)\right) = +\infty.} \tag{52.5}$$

This is the exact position of Twin in the unified infinite architecture.

Chapter 53 General Fixed Differences

For any fixed

$$h \geq 1,$$

define

$$\Gamma^{[h]} = \{r: \delta(r) = h\}.$$

The corresponding ordinary prime difference is

$$2h.$$

A persistent relation sieve for a general fixed difference can be built in the following order:

1. construct an adapted square horizon;
2. define the old-history relation supply;
3. identify the death form at the final control layer;
4. establish a finite absorption resource pool;
5. compute the pattern absorption reuse degree;
6. obtain a window-by-window capacity upper bound;
7. define the net relation potential;
8. accumulate the net potential;
9. apply the infinite-recurrence criterion.

Therefore

Twin is the minimal model of a persistent relation sieve for general fixed differences.

(53.1)

Chapter 54 General Prime Tuples

For

$$H = \{h_1, \dots, h_k\},$$

define the relation

$$R_H(n) = \prod_{j=1}^{k} \alpha\,(n + h_j).$$

First test for local obstruction:

$$\nu_H(q) = q.$$

If this occurs, infinite recurrence is closed off by the local structure, leaving only finitely many boundary exceptions.

If the pattern passes local admissibility, proceed through

$$\boxed{B_H \to U \to \mu_H \to C_H \to e_H \to \mathcal{E}_H.} \tag{54.1}$$

Thus infinite recurrence for a general tuple receives the same standard interface as Twin.

Figure 5-10 Supply–Absorption Framework for a General Fixed Pattern H

Twin is only the minimal case H={0,2}; a general pattern uses three execution blocks: pattern control → horizon ledger → capacity and accumulation.

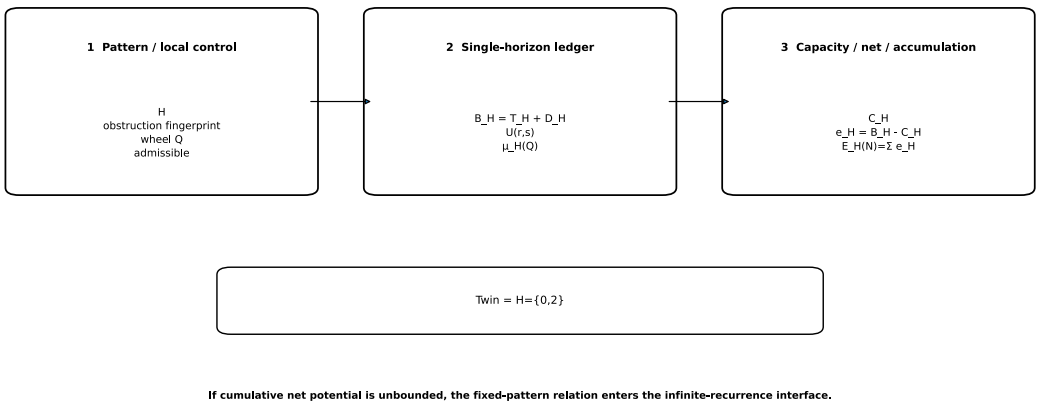


Figure 5-10. Supply–Absorption Framework for a General Fixed Pattern H. Twin with $H = \{0,2\}$ is only the minimal instance of a general fixed-offset pattern. After local admissibility and the wheel Q, the pattern enters the common supply–absorption chain $B_H = T_H + D_H$, resource pool $U(r,s)$, reuse degree $\mu_H(Q)$, capacity C_H, net potential e_H, and cumulative quantity $\mathcal{E}_H$; unbounded cumulative net potential then enters the infinite-recurrence interface for the fixed pattern.

Part VII Batch Structures for Other Infinite Interfaces

Chapter 55 Sophie Germain

Define

$$\boxed{SG(p) = \alpha(p)\alpha(2p+1).} \tag{55.1}$$

Its infinite form is

$$\boxed{\exists^{\infty} p : SG(p) = 1.} \tag{55.2}$$

Hence the standard interface asks whether

$$\boxed{C_{SG}(X) = \sum_{p \le X} S\,G(p)}$$

is unbounded.

Like Twin, this belongs to

INF_RECURRENCE,

but the position maps differ, so the finite-control geometry and absorption structure differ as well.

Chapter 56 Mersenne

For prime exponent p, define

$$\boxed{M(p) = \alpha(2^p - 1).} \tag{56.1}$$

If one asks whether there are infinitely many Mersenne primes, then

$$\boxed{C_M(X) = \sum_{\substack{p \le X \\ p \in \mathbb{P}}} M(p)} \tag{56.2}$$

must be unbounded.

Here relation positions grow rapidly along the exponential map

$$p \mapsto 2^p - 1,$$

so ordinary square-pattern windows are unsuitable, but the quantifier interface remains the same:

$$\boxed{\text{INF_RECURRENCE.}}$$

Thus

what is unified is the infinite criterion;

the finite certification tool may vary with the relation orbit.

(56.3)

Chapter 57 Bunyakovsky and Schinzel H

Given an integer-valued polynomial

$$f(n),$$

define

$$R_f(n) = \alpha(f(n)).$$

The infinite form remains

$$\exists^{\infty} n : R_f(n) = 1,$$

that is,

$$\sum_{n \le X} R_f(n)$$

is unbounded.

For a polynomial family

$$f_1, \ldots, f_k,$$

define

$$R_{\boldsymbol{f}}(n) = \prod_j \alpha(f_j(n)).$$

First check complete local coverage:

Does there exist a prime q such that every input residue class forces at least one $f_j(n) \equiv 0 \pmod{q}$?

If so, infinite all-prime recurrence is locally obstructed.

Otherwise one proceeds to the global infinite interface.

Chapter 58 The Arbitrarily-Long Interface of Green–Tao

Define

$$C_k(a,h) = \prod_{j=0}^{k-1} \alpha\,(a + jh).$$

Its quantifier is

$$\forall k\ \exists a,h\colon C_k(a,h) = 1.$$

This is not infinite recurrence for a fixed k.

It requires

$$\boxed{L(X) = \max\{k\colon \exists a,h \text{ realized within horizon } X\}}$$

to be unbounded.

Thus

$$\boxed{\text{ARBITRARILY_LONG}}$$

has its own standard criterion.

Chapter 59 Goldbach, Legendre, and Universal-Type Interfaces

Goldbach,

$$\forall N\ \exists p: R_N(p) = 1,$$

belongs to

UNIVERSAL_FIBER.

Legendre,

$$\forall n\ \exists p \in (n^2, (n+1)^2),$$

belongs to

WINDOW_UNIVERSAL.

Their distinction from the persistent non-extinction type can be compressed into

universal type	infinite-recurrence type
every parameter must succeed	the number of successes must be unbounded
no fiber may be missed	arbitrarily large gaps are allowed
target: positive lower bound for each fiber	target: unbounded cumulative quantity

(59.1)

Chapter 60 Brocard, Andrica, Firoozbakht, and Grimm

These problems have quantifier form

$$\forall n, \quad P(n).$$

Hence the standard failure count is

$$\boxed{F(X) = \sum_{n \le X} \mathbf{1}_{\{\neg P(n)\}}.} \tag{60.1}$$

The universal proposition is equivalent to

$$\boxed{F(X) = 0 \quad \forall X.} \tag{60.2}$$

For problems of this kind, increasing a success count is irrelevant.

What must be controlled is

$$\boxed{\text{whether a failure ever occurs.}}$$

Chapter 61 Erdős–Woods and Identity-Block Separation

Define the factor-identity block

$$B_K(x) = (\mathrm{PF}(x), \ldots, \mathrm{PF}(x + K - 1)).$$

Such problems do not ask whether a relation occurs infinitely often, but rather:

> How long a complete factor-identity block is sufficient to distinguish two different starting points?

Thus the standard interface is to find

$$\boxed{K_*}$$

such that

$$x \neq y \Rightarrow B_{K_*}(x) \neq B_{K_*}(y).$$

This is

$$\boxed{\text{BLOCK_SEPARATION.}}$$

Part VIII Standard Structures for Batch Proofs

Chapter 62 Batch Proof Is Not Batch Computation

A finite computation may process

$$10^3,\ 10^6,\ 10^9$$

instances at once.

But as long as the parameter set remains finite, the result is still only a finite fact.

A batch proof requires

> the same rule to apply to an arbitrary parameter.

Therefore a complete batch certificate must simultaneously satisfy

$$\boxed{\text{relation fidelity} + \text{local correctness} + \text{parameter coverage} + \text{quantifier closure.}} \tag{62.1}$$

Part III has already put propositions into standard relational-quantifier form; Part IV has sieved finite relations into terminal identities; this part provides certificate functions that enter the quantifier layer directly.

Thus

$$\boxed{\text{normal form of a classical proposition} \Leftarrow \text{quantifier closer} \Leftarrow \text{family of finite certificates.}} \tag{62.2}$$

Chapter 63 Three Standard Closures for Infinite Recurrence

63.1 Positive certificate window by window

If infinitely many windows satisfy

$$e_j > 0,$$

then infinitely many windows contain genuine relations.

63.2 Cumulative net potential

If

$$\limsup_{N} \sum_{j \le N} e_j = +\infty,$$

then the cumulative number of genuine relations is unbounded.

63.3 Direct cumulative lower bound

If there exists an unbounded function

$$L(N)$$

such that

$$\sum_{j \le N} T_j \ge L(N),$$

then infinite recurrence holds.

The three forms, from stronger to weaker, have different ranges of applicability.

Chapter 64 Three Standard Closures for Universal-Type Propositions

64.1 Uniform positive lower bound

$$C(\theta) \geq L(\theta) > 0.$$

64.2 Finite-quotient coverage

If all parameters fall into finitely many state classes and every state class guarantees nonemptiness, then universality follows.

64.3 Exclusion of the least failure

If the parameters are well ordered, assume a least failing parameter exists, and use the historical structure to force that parameter to succeed as well; then the failure set is empty.

Chapter 65 Standard Closure for Arbitrarily Long Relations

To prove

$$\forall k \ \exists \Theta, x,$$

it is enough to construct a certificate valid for every k.

Equivalently, one may prove that

$$L(X)$$

is unbounded.

There is no requirement that the relation for a fixed k occur infinitely often.

Therefore

arbitrary length and infinite recurrence are two orthogonal directions.

(65.1)

Chapter 66 Standard Closure for Parameter Infinitude

If the parameters themselves are

$$\theta_1, \theta_2, \ldots,$$

define the cumulative number of successful parameters by

$$P(N).$$

As soon as

$$P(N)$$

is unbounded, parameter infinitude follows.

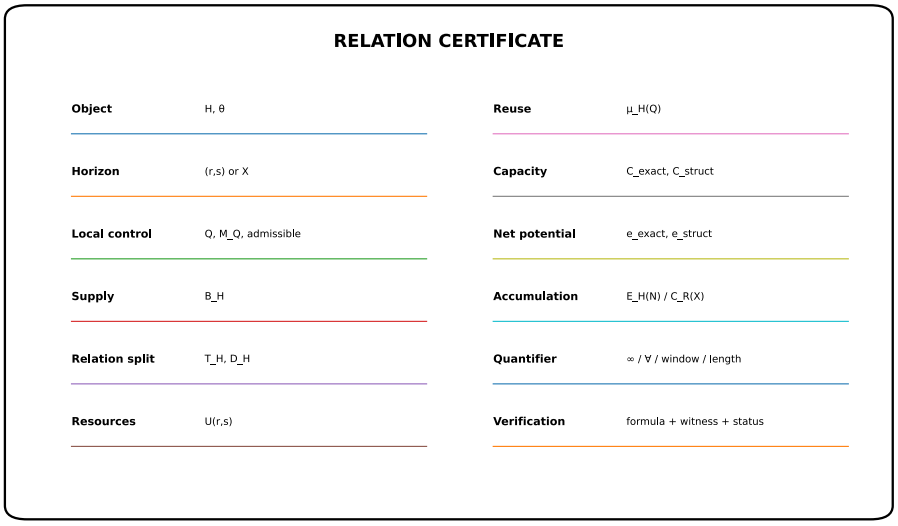
This is appropriate for situations where

each successful parameter requires only one witness, but infinitely many successful parameters are required.

Part IX Finite Certificates at the Relation Level

Figure 5-11 Relation-Level Certificate Data Structure

The certificate record keeps the proof object, finite horizon, relation supply, death responsibility, absorption capacity, net potential, and global quantifier in one auditable structure.



Certificate granularity should match relation-object granularity; machine verification checks the certificate and does not replace theorem-level quantifier closure.

Figure 5-11. Relation-Level Certificate Data Structure. A relation-level certificate organizes the pattern or relation parameters, finite horizon, wheel and local admissibility, supply B_H, genuine/death split T_H, D_H, resource pool U, reuse degree μ_H, exact/structural capacities, net potential, cumulative quantity, quantifier interface, and verification status in one record. Certificate granularity should match relation-object granularity so that human review and machine verification inspect the same mathematical object.

Chapter 67 Relation Objects and Certificate Objects Should Have the Same Granularity

For a given enormous unit-difference candidate

$$(M-1, M+1),$$

if the proof object is the entire relation, then the most natural certificate should likewise share, as far as possible,

- the same center;
- the same modulus;
- the same Lucas system;
- the same group-order constraints.

Therefore

the granularity of relation certification should match the granularity of the relation object itself.

(67.1)

Chapter 68 General Single-Relation Certificate for Twin

Let

$$M > 4$$

be even.

Define

$$\boxed{K = (M-1)(M+1) = M^2 - 1.} \tag{68.1}$$

Take the Lucas sequence

$$U_0 = 0, \qquad U_1 = 1,$$

$$\boxed{U_{n+2} = P\,U_{n+1} - Q\,U_n.} \tag{68.2}$$

with discriminant

$$D = P^2 - 4Q.$$

Theorem 68.1 General Single-Relation Certificate for Twin

Assume the complete prime factorization of M is known and that

$$\boxed{\gcd(K, 2QD) = 1,} \tag{68.3}$$

$$\boxed{U_M(P,Q) \equiv 0 \pmod{K},} \tag{68.4}$$

and for every distinct prime factor q of M,

$$\boxed{\gcd(U_{M/q}(P,Q), K) = 1.} \tag{68.5}$$

Then

$$\boxed{M-1, \qquad M+1}$$

are both prime.

Proof

Take any prime divisor

$$\ell \mid K.$$

From

$$U_M \equiv 0 \pmod{\ell}$$

we know that the rank of apparition of the Lucas sequence satisfies

$$\rho_\ell \mid M.$$

Since

$$\ell \nmid U_{M/q}$$

for every distinct prime factor $q \mid M$, we obtain

$$\rho_\ell = M.$$

The Lucas rank of apparition satisfies

$$\rho_\ell \mid \ell - \left(\frac{D}{\ell}\right).$$

Therefore

$$M \mid \ell \pm 1.$$

If

$$\ell \mid M - 1,$$

then

$$\ell \leq M - 1.$$

Together with

$$\ell \equiv \pm 1 \pmod{M},$$

the only possibility is

$$\ell = M - 1.$$

Thus

$$M - 1$$

is prime.

If

$$\ell \mid M + 1,$$

then similarly the only candidates are

$$M - 1, \quad M + 1.$$

But

$$\gcd(M - 1, M + 1) = 1,$$

so an odd prime divisor of $M + 1$ cannot simultaneously equal $M - 1$.

Hence

$$\ell = M + 1.$$

Thus

$$M + 1$$

is also prime. This proves the theorem.

Chapter 69 Certificate from One Sufficiently Thick Principal Chain

Suppose

$$\boxed{M = k2^e,} \tag{69.1}$$

where k is a positive odd integer.

Let

$$F = 2^e,$$

$$N_- = M - 1, \qquad N_+ = M + 1.$$

Take the Lucas V-sequence

$$V_0 = 2, \qquad V_1 = P,$$

$$\boxed{V_{n+2} = PV_{n+1} - QV_n.} \tag{69.2}$$

Theorem 69.1 Twin Certificate from One Sufficiently Thick Principal Chain

If

$$\boxed{F > k + 2,} \tag{69.3}$$

$$\boxed{\gcd(N_-N_+, 2QD) = 1,} \tag{69.4}$$

and

$$\boxed{V_{M/2}(P, Q) \equiv 0 \pmod{N_-},} \tag{69.5}$$

$$\boxed{V_{M/2}(P, Q) \equiv 0 \pmod{N_+},} \tag{69.6}$$

then

$$\boxed{M - 1, \qquad M + 1}$$

are both prime.

Proof

Take any prime divisor ℓ of either N_- or N_+.

From

$$V_{M/2} \equiv 0 \pmod{\ell}$$

the corresponding ratio z of characteristic roots satisfies

$$z^{M/2} = -1, \qquad z^M = 1.$$

If its multiplicative order is R_ℓ, then

$$R_\ell \mid M, \qquad R_\ell \nmid M/2.$$

Since

$$M = k2^e, \qquad k \text{ is odd,}$$

we obtain

$$2^e \mid R_\ell.$$

The Lucas group-order constraint gives

$$R_\ell \mid \ell - \left(\frac{D}{\ell}\right).$$

Hence

$$\ell \geq F - 1.$$

If $N_\pm$ were composite, it would have at least two prime factors counted with multiplicity, so

$$N_\pm \geq (F-1)^2.$$

On the other hand,

$$N_\pm \leq kF + 1.$$

But

$$F > k + 2$$

implies

$$kF + 1 < (F-1)^2,$$

a contradiction.

Therefore

$$M - 1, \qquad M + 1$$

are both prime. This proves the theorem.

Consequently,

there is no need to recover the complete factor history of the center; once one common principal (69.7)

Chapter 70 Moving Horizons and Relation Certificates Are the Same Principle at Two Scales

A square horizon answers:

> *How can a batch of candidates be completely and finitely certified?*

A Lucas relation certificate answers:

> *How can one enormous candidate be completely and finitely certified?*

The two share the same principle:

once finite information has crossed every threshold capable of changing the target identity,
later information can no longer rewrite the conclusion.

(70.1)

Thus finite horizons and finite certificates are not two unrelated techniques.
They are manifestations, at different scales, of the same terminal principle.

Part X Finite Computational Verification

Chapter 71 905 Consecutive Square Horizons

Take

$$s^2 < 5 \times 10^7.$$

The consecutive odd-prime triples

$$p < r < s$$

yield

$$\boxed{905}$$

square horizons.

For each window compute

$$B_{r,s}, \qquad |U_{r,s}^{\text{full}}|, \qquad T_{r,s}, \qquad D_{r,s}.$$

The inequality

$$\boxed{B_{r,s} \geq |U_{r,s}^{\text{full}}|}$$

holds in all 905 horizons.

Equality occurs at

$$\boxed{r = 11,\ 17,\ 29.}$$

The remaining 902 horizons satisfy strictly

$$B > |U^{\text{full}}|.$$

Cumulatively,

$$\boxed{\sum B = 239\,213,}$$

$$\boxed{\sum T = 238\,962,}$$

$$\boxed{\sum D = 251,}$$

$$\boxed{\sum |U^{\text{full}}| = 2\,546.}$$

For the final verified horizon,

$$p = 7043, \qquad r = 7057, \qquad s = 7069,$$

we obtain

$$B = 657, \qquad |U^{\text{full}}| = 3, \qquad T = 657, \qquad D = 0.$$

These finite data verify, item by item, the square-horizon formulas, the injection of deaths into resources, and the actual scale of the resource pool.

Chapter 72 Structural Net Potential on the Same Finite Data

For the unit-difference pattern

$$H = \{0,2\}, \qquad \mu_H = 1,$$

the structural capacity is

$$C^{\mathrm{struct}} = \Delta - 1.$$

Over the same 905 windows,

$$\boxed{\sum(\Delta - 1) = 6\,159.}$$

Hence the cumulative structural net potential is

$$\boxed{\sum e^{\mathrm{struct}} = 239\,213 - 6\,159 = 233\,054.} \tag{72.1}$$

Within this finite collection of windows, the minimum structural net potential is positive, and the smallest window margin is

$$\boxed{1}.$$

Thus the unified structural capacity still provides a relation certificate window by window throughout this finite range.

Chapter 73 Finite Verification of Historical Resource Reuse

For all primes

$$u < 2 \times 10^6,$$

compute the number of square horizons for which

$$s \le u < \frac{s^2}{r}.$$

The result is

$$\boxed{\max_{u<2\times10^6} \rho(u) = 5.}$$

The distribution of nonzero eligibility counts is:

Eligibility count	Number of atoms
1	47,861
2	72,895
3	25,519
4	2,570
5	86

If we count only actual death witnesses satisfying

$$ru + 2\varepsilon_r(u) = v \in \mathbb{P},$$

then the maximum number of successful reuses of the same u is

$$\boxed{3}.$$

The distribution is:

Actual death count	Number of atoms
1	24,576
2	1,271
3	21

This agrees with the resource half-life and finite-lifetime theorems.

Chapter 74 Computational Comparison of Relation-Level Certificates

For a relation instance of about 380 digits, using the same fast Lucas-recursion kernel:

Method	Verification time
Separate verification of the two endpoints	3.782 s
General relation certificate with a common modulus	8.251 s
Relation certificate decomposed by the two endpoints	4.855 s

Thus

logical integration does not automatically imply faster runtime.

(74.1)

For a structured center of about 1040 digits,

$$256200945 \cdot 2^{3426} \pm 1,$$

one set of parameters for the complete relation certificate is

$$P = 81, \qquad Q = 1641, \qquad D = -3.$$

The recorded times are

$$T_{\text{search}} = 7.799 \text{ s},$$

$$T_{\text{verify}} = 1.112 \text{ s}.$$

After retaining only the sufficiently thick principal chain 2^{3426}, one set of parameters is

$$P = 27, \qquad Q = 183, \qquad D = -3.$$

The recorded times are

$$T_{\text{search}} = 0.423 \text{ s},$$

$$T_{\text{verify}} = 0.0378 \text{ s}.$$

In this implementation, the corresponding differences are approximately

$$18.45 \times$$

in witness search and

$$29.4 \times$$

in verification.

The mathematical conclusion is

the structure required for verification can be compressed from "complete factorization of the center" to "one sufficiently thick common principal chain."

(74.2)

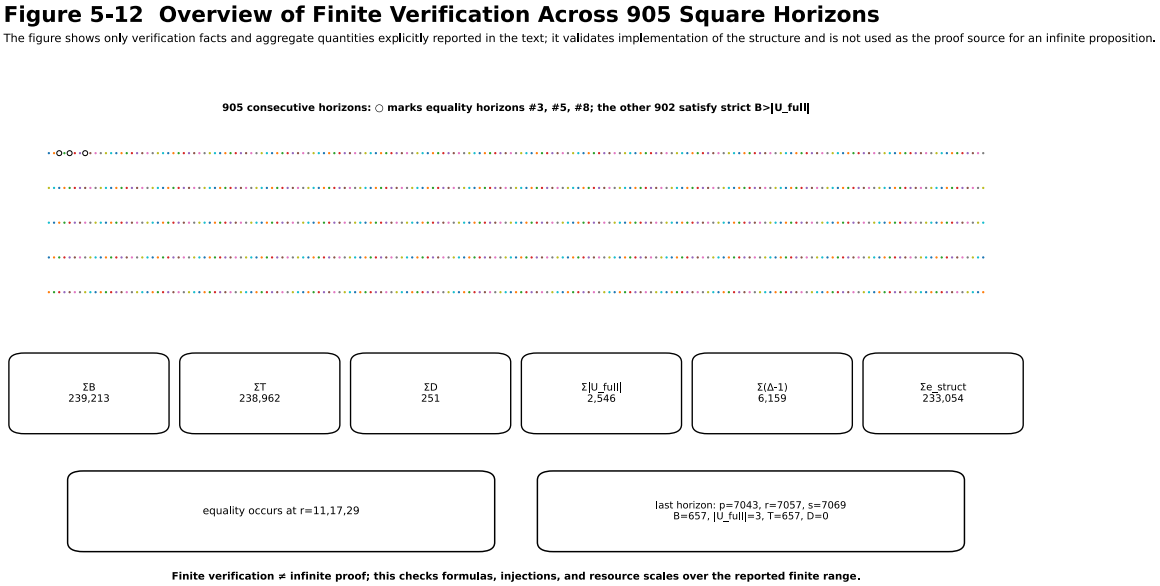


Figure 5-12. Overview of Finite Verification Across 905 Square Horizons. The text verifies $B_{r,s}, |U_{r,s}^{\text{full}}|, T_{r,s}, D_{r,s}$ window by window over 905 consecutive square horizons with $s^2 < 5 \times 10^7$. Every horizon satisfies $B \geq |U^{\text{full}}|$; equality occurs at $r = 11,17,29$, while the other 902 are strict. The aggregates are $\sum B = 239213, \sum T = 238962, \sum D = 251, \sum |U^{\text{full}}| = 2546$, and the same finite dataset gives $\sum e^{\text{struct}} = 233054$. These data verify the finite structural implementation and do not replace the quantifier proof of an infinite proposition.

Part XI Parameter Dictionary

Chapter 75 Meaning of the Parameters in the Infinite-Relation Module

To keep the formulas, web implementation, and computational implementation on a single convention, this part fixes the following parameters.

75.1 H: fixed offset pattern

$$H = \{h_1, \dots, h_k\}.$$

It determines which translated positions are read by one local relation instance.

75.2 $p < r < s$: consecutive odd primes

- p: largest control prime in the old history;
- r: current final control prime;
- s: next control prime.

75.3 $\Delta = s - r$: current prime gap

It simultaneously controls

- the width of the square window;
- the width of the absorption-resource window;
- the structural capacity upper bound.

75.4 B_H: relation supply

The number of complete patterns that survive every old historical layer.

75.5 T_H: genuine relations

The number of patterns whose endpoints are all genuine primes.

75.6 D_H: genuine deaths

$$D_H = B_H - T_H.$$

75.7 $U(r, s)$: absorption resource pool

$$U(r, s) = \{u \in \mathbb{P} : s \le u < s^2/r\}.$$

75.8 Q: set of wheel control primes

Used to compute how many pattern roles a single resource can absorb repeatedly.

75.9 M_Q: wheel modulus

$$M_Q = \prod_{q \in Q} q.$$

75.10 $\mu_H(Q)$: absorption reuse degree

The maximum number of pattern endpoint roles that the same resource can support under the finite wheel conditions.

75.11 C_H^{exact}: fixed-spectrum resource capacity

$$C_H^{\mathrm{exact}} = \mu_H(Q)|U|.$$

75.12 C_H^{struct}: structural capacity

$$C_H^{\mathrm{struct}} = \mu_H(Q)(\Delta - 1).$$

75.13 e_H: net relation potential

$$e_H = B_H - C_H.$$

It means

> *the current relation supply minus a provable maximal absorption capacity.*

75.14 $\mathcal{E}_H$: cumulative net relation potential

$$\mathcal{E}_H(N) = \sum_{j \le N} e_{H,j}.$$

It is the central cumulative quantity by which finite-horizon certificates enter infinite recurrence.

Chapter 76 Meaning of the Parameters in the Quantifier Module

$$X$$

A finite horizon or cumulative upper bound.

$$\theta$$

A relation parameter, such as

- a fixed sum;
- a fixed difference;
- a polynomial;
- a window index;
- a block starting point.

$$x$$

The relation anchor.

$$k$$

A relation length, pattern size, or window block length.

$$C_R(X)$$

Cumulative hit count for infinite recurrence.

$$C(\theta)$$

Number of witnesses for a fixed parameter in a universal fiber.

$$W(n)$$

Window occupancy count.

$$L(X)$$

Maximum relation length realized within a finite horizon.

$$P(X)$$

Cumulative number of successful parameters.

$$F(X)$$

Cumulative failure count for an indexwise universal problem.

Part XII Finite Certificates Enter Formal Proofs of Classical Propositions

Chapter 77 Certificates as Theorem Interfaces

After a classical proposition has been normalized relationally in Part III, it has the standard form

$$\boxed{Q_{\theta,x}[R_\theta(x) = 1].} \tag{77.1}$$

The certificates of Part V do not change the proposition; they provide finite lower bounds, cumulative lower bounds, or exclusion statements that can enter the quantifier layer directly.

For example, the standard count for a universal fiber is

$$C(\theta) = \sum_x R_\theta\,(x).$$

If

$$\boxed{C(\theta) \geq L(\theta) > 0,} \tag{77.2}$$

then

$$\boxed{\exists x: R_\theta(x) = 1.} \tag{77.3}$$

If this holds for every parameter, then

$$\boxed{\forall\theta\ \exists x: R_\theta(x) = 1.} \tag{77.4}$$

Theorem 77.1 Certificate Transfer for Universal Fibers

If

$$C(\theta) \geq L(\theta) > 0 \quad \forall\theta,$$

then

$$\boxed{\forall\theta\ \exists x: R_\theta(x) = 1.} \tag{77.5}$$

Proof

For arbitrary θ, $C(\theta) > 0$ means that at least one relation instance is a hit. Since θ was arbitrary, the universal statement follows. This proves the theorem.

77.2 Certificate transfer for infinite recurrence

Let

$$C_R(X) = \sum_{x \leq X} R\,(x).$$

If

$$\boxed{C_R(X) \geq E(X),} \tag{77.6}$$

and

$$\boxed{\limsup_{X\to\infty} E(X) = +\infty,} \tag{77.7}$$

then $C_R(X)$ is unbounded.

Therefore:

Theorem 77.2 Certificate Transfer for Infinite Recurrence

$$\boxed{C_R(X) \geq E(X), \quad \text{limsup} E(X) = +\infty \Rightarrow \exists^{\infty} x : R(x) = 1.} \tag{77.8}$$

This is precisely the abstract form of the cumulative-net-relation-potential theorem.

77.3 Certificate entries for the other quantifiers

Window universality uses

$$W(n) \geq 1 \quad \forall n.$$

Arbitrary length uses

$$L(X) \text{ unbounded.}$$

Parameter infinitude uses

$$P(X) \text{ unbounded.}$$

Indexwise universality uses

$$F(X) = 0 \quad \forall X.$$

Identity-block separation uses a uniform finite block length K.

Thus all seven quantifier classes of Part III have explicit certificate entries in Part V.

Chapter 78 Formal Proof and Infinite Closure of Twin

The classical Twin proposition has the lossless chain

$$\boxed{\begin{aligned} &\exists^{\infty} p:\ p, p+2 \in \mathbb{P} \\ \Leftrightarrow\ &\exists^{\infty} a:\ \hat{\alpha}(a)\hat{\alpha}(a+1) = 1 \\ \Leftrightarrow\ &\Gamma^{[1]} = \infty \\ \Leftrightarrow\ &C_T(X) \text{ is unbounded} \\ \Leftrightarrow\ &\forall K\ \exists j > K:\ T_j > 0. \end{aligned}} \tag{78.1}$$

The single-horizon supply–absorption theorem gives

$$\boxed{T_j \geq B_j - |U_j| \geq B_j - (\Delta_j - 1).} \tag{78.2}$$

Define

$$e_j^{\text{exact}} = B_j - |U_j|, \qquad e_j^{\text{struct}} = B_j - (\Delta_j - 1),$$

and cumulative quantities

$$\boxed{\mathcal{E}_T^{\text{exact}}(N) = \sum_{j \leq N} e_j^{\text{exact}},} \tag{78.3}$$

$$\boxed{\mathcal{E}_T^{\text{struct}}(N) = \sum_{j \leq N} e_j^{\text{struct}}.} \tag{78.4}$$

Then

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_T^{\text{exact}}(N) = +\infty \Rightarrow \exists^{\infty} p:\ p, p+2 \in \mathbb{P},} \tag{78.5}$$

and

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_T^{\text{struct}}(N) = +\infty \Rightarrow \exists^{\infty} p:\ p, p+2 \in \mathbb{P}.} \tag{78.6}$$

Thus the Twin certificate chain is

$$\boxed{\text{single-horizon terminal certification} \rightarrow \text{supply–absorption lower bound} \rightarrow \text{cumulative net poten}} \tag{78.7}$$

For the unit-difference pattern

$$H = \{0,2\}, \qquad \mu_H = 1,$$

the persistent double-escape supply, the single-resource absorption upper bound, resource half-life, and finite lifetime established above together complete the cumulative supply–cumulative absorption comparison and yield an unbounded cumulative net potential. Substitution into Theorem 77.2 (equivalently, Theorem 46.1) gives

$$\boxed{\exists^{\infty} p:\ p, p+2 \in \mathbb{P}.} \tag{78.7a}$$

Therefore equations (78.1)–(78.7a) constitute the complete proof chain from finite moving horizons to infinite recurrence for Twin.

The relation-level Lucas certificate operates at another scale: it provides direct finite terminal certification for one specified enormous Twin candidate.

Thus

$$\boxed{\text{square-horizon certificates handle relations in batches; relation-level Lucas certificates handle o}} \tag{78.8}$$

Figure 5-09 Twin as a Persistent Non-Extinction Relation

Consecutive square horizons provide finite zero-error relation certificates; the supply–absorption ledger accumulates into net potential and enters the infinite-recurrence quantifier.

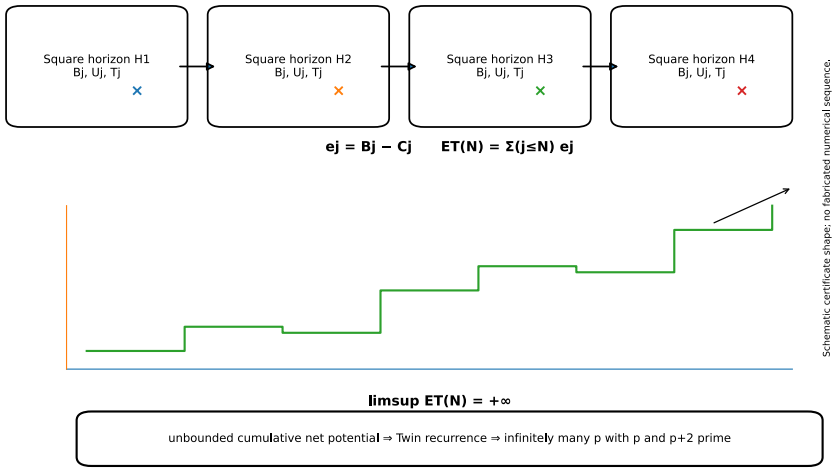


Figure 5-09. Twin as a Persistent Non-Extinction Relation. Twin does not rely on a common terminal horizon. Across consecutive square horizons, it uses single-horizon terminal certification, supply–absorption comparison, and cumulative net-potential accounting. Unbounded cumulative net potential is then read as infinite recurrence of the unit-gap relation.

Chapter 79 How General Classical Propositions Invoke Part V

79.1 Fixed prime tuple

For

$$H = \{h_1, \dots, h_k\},$$

the relation is

$$R_H(n) = \prod_{j=1}^{k} \alpha\,(n + h_j).$$

After the pattern passes the local-obstruction test, this part constructs

$$B_H, \quad U, \quad \mu_H, \quad C_H, \quad e_H, \quad \mathcal{E}_H.$$

If

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_H(N) = +\infty,}$$

then the pattern recurs infinitely often.

79.2 Goldbach-type universal fibers

Goldbach belongs to

UNIVERSAL_FIBER.

Part IV has already established terminal freezing for fixed sums and the equivalence with relation completeness.

If for every target m one has

$$\boxed{G(m) \geq L(m) > 0,} \tag{79.1}$$

or a uniform coverage certificate is established on the natural terminal segment, then

$$\boxed{\forall m \geq 3:\ G(m) > 0.} \tag{79.2}$$

79.3 Window universality

If

$$\boxed{W(n) \geq 1} \tag{79.3}$$

holds for every window parameter, then window universality follows.

79.4 Arbitrary length

A Green–Tao-type proposition uses

$$\boxed{L(X) \text{ unbounded.}} \tag{79.4}$$

79.5 Parameter infinitude

A Mersenne-type parameter count can be written as

$$\boxed{P(X) = \#\{p \le X : 2^p - 1 \in \mathbb{P}\}.} \tag{79.5}$$

Unboundedness of $P(X)$ yields infinitely many successful parameters.

79.6 Unified invocation chain

The proof interface for a classical proposition can therefore be written as

$$\boxed{\text{classical proposition} \Leftrightarrow \text{relational normal form} \Leftarrow \text{family of finite certificates} \Leftarrow \text{finite-control th}} \tag{79.6}$$

Here

- Part III provides the relational normal form;
- Part IV provides finite control and terminal identities;
- Part V provides certificates and their transfer through quantifiers;
- Part VI completes the two-way formalization and certificate invocation for the individual classical propositions.

Therefore

Part V is the global certificate library directly callable by the formal proofs of classical propositions.

(79.7)

Part XIII Table of Core Theorems in This Part

Chapter 80 Infinite-Quantifier Theorems

T1 Counting Criterion for Infinite Recurrence

$$\boxed{\exists^{\infty}x: R(x) = 1 \Leftrightarrow C_R(X) \text{ is unbounded.}}$$

T2 Universal-Fiber Criterion

$$\boxed{\forall\theta\exists x: R_\theta(x) = 1 \Leftrightarrow C(\theta) > 0 \quad \forall\theta.}$$

T3 Window-Universality Criterion

$$\boxed{\forall n\exists x \in I_n \Leftrightarrow W(n) \geq 1 \quad \forall n.}$$

T4 Arbitrary-Length Criterion

$$\boxed{\forall k\exists\Theta, x \Leftrightarrow L(X) \text{ is unbounded.}}$$

T5 Parameter-Infinitude Criterion

$$\boxed{\exists^{\infty}\theta\exists x \Leftrightarrow P(X) \text{ is unbounded.}}$$

Chapter 81 Persistent Relation-Sieve Theorems

T6 Exact Twin Formula in a Square Horizon

$$\boxed{\mathcal{T}_{r,s} = \{n: r^2 \le n < n+2 < s^2,\ \gcd(n(n+2), r^{\#}) = 1\}.}$$

T7 One Cofactor, One Death

For fixed $r, u > 3$, one cofactor can absorb at most one unit-difference relation in a single horizon.

T8 Clean Zone

If an old double-escape relation lies entirely in

$$(r^2, rs),$$

then it is directly a Twin.

T9 Resource Half-Life

$$\boxed{\log(u/s) < \frac{1}{2}\log(u/r).}$$

T10 Finite Resource Lifetime

$$\boxed{\rho(u) < \infty.}$$

Chapter 82 Supply–Absorption Theorems for General Patterns

T11 Single-Layer Purification

An interior composite endpoint that survives the old history must be of the form

$$\boxed{ru, \qquad u \in U(r,s).}$$

T12 Absorption Capacity

$$\boxed{D_H \le \mu_H(Q)|U|.}$$

T13 Structural Resource Upper Bound

$$\boxed{|U| \le \Delta - 1 \qquad (r \ge 5).}$$

T14 Exact Net-Potential Lower Bound

$$\boxed{T_H \ge e_H^{\text{exact}}.}$$

T15 Structural Net-Potential Lower Bound

$$\boxed{T_H \ge e_H^{\text{struct}}.}$$

T16 Infinite Criterion from Cumulative Exact Net Potential

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_H^{\text{exact}}(N) = +\infty \Rightarrow H\text{-relations recur infinitely often.}}$$

T17 Infinite Criterion from Cumulative Structural Net Potential

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_H^{\text{struct}}(N) = +\infty \Rightarrow H\text{-relations recur infinitely often.}}$$

Chapter 83 Relation-Level Certificate Theorems

T18 General Single-Relation Twin Certificate

If

$$\gcd(K, 2QD) = 1,$$

$$U_M(P, Q) \equiv 0 \pmod{K},$$

and

$$\gcd(U_{M/q}, K) = 1$$

for every distinct prime factor $q \mid M$, then

$$M - 1,\ M + 1$$

are both prime.

T19 Certificate from One Sufficiently Thick Principal Chain

If

$$M = k2^e,$$

the common 2-power principal chain satisfies

$$2^e > k + 2,$$

and the corresponding Lucas V-sequence group-order conditions hold, then

$$M - 1,\ M + 1$$

are both prime.

T20 Certificate Transfer for Universal Fibers

If

$$C(\theta) \geq L(\theta) > 0 \quad \forall \theta,$$

then

$$\boxed{\forall \theta\ \exists x : R_\theta(x) = 1.}$$

T21 Certificate Transfer for Infinite Recurrence

If

$$C_R(X) \geq E(X), \qquad \limsup_{X \to \infty} E(X) = +\infty,$$

then

$$\boxed{\exists^{\infty} x : R(x) = 1.}$$

T22 Twin Certificate from Cumulative Net Potential

$$\boxed{\limsup_{N\to\infty}\mathcal{E}_T^{\mathrm{exact}}(N) = +\infty \Rightarrow \text{Twin recurs infinitely often.}}$$

and

$$\boxed{\limsup_{N\to\infty}\mathcal{E}_T^{\mathrm{struct}}(N) = +\infty \Rightarrow \text{Twin recurs infinitely often.}}$$

T23 Certificate Interface for Classical Propositions

Once a classical proposition has been written as a standard relational quantifier, the corresponding finite certificate and quantifier closer from Part V can enter its formal proof directly.

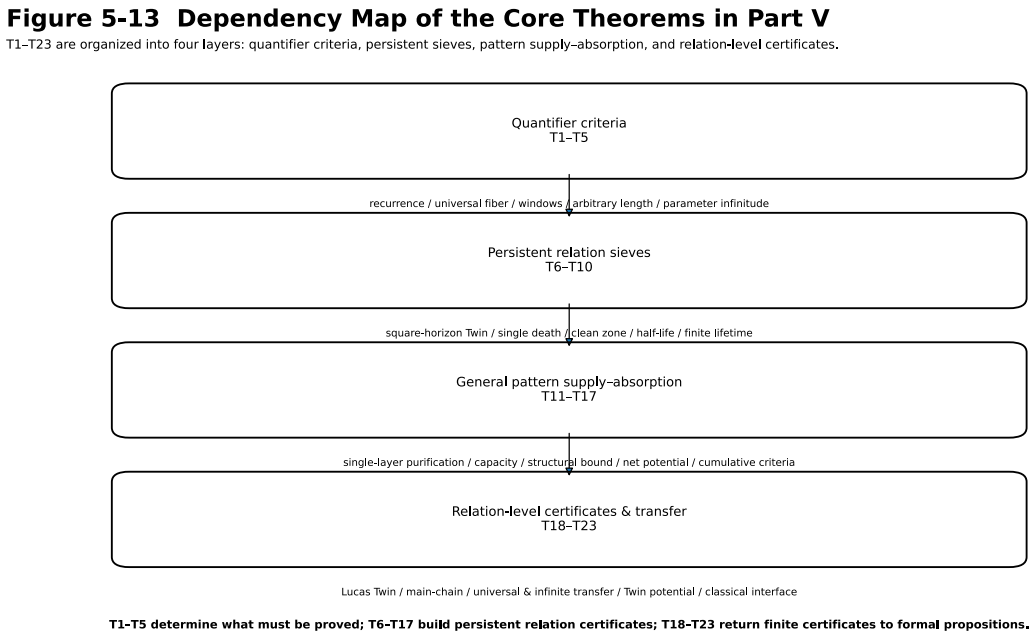


Figure 5-13. Dependency Map of the Core Theorems in Part V. Theorems T1–T23 of Part V fall into four layers: T1–T5 give the standard finite criteria for the different global quantifiers; T6–T10 establish square horizons, single death, the clean zone, resource half-life, and finite lifetime for persistent relation sieves; T11–T17 establish purification, absorption capacity, structural bounds, net potential, and cumulative infinitude criteria for general patterns; T18–T23 return Lucas relation certificates, main-chain certificates, and universal/infinite certificate transfer to formal classical propositions.

Conclusion Infinity Is the Unbounded Accumulation of Finite Certificates

An infinite relation is easily phrased as a vague question:

> *Will this relation keep appearing forever?*

After decomposing that question, this part obtains a completely finite internal chain.

First choose a finite square horizon.

Within this horizon, every endpoint identity can be terminally certified.

The old history leaves

$$B_H$$

complete units of relation supply.

If one unit of supply ultimately dies, it must leave a genuine absorption witness of the form

$$ru.$$

Resources can come only from the finite set

$$U(r,s).$$

The internal structure of the pattern further restricts how many roles the same resource can play:

$$\mu_H(Q).$$

Thus the total number of deaths cannot exceed

$$\mu_H(Q)|U|.$$

Hence the current window retains at least

$$\boxed{B_H - \mu_H(Q)|U|}$$

units of net relation potential.

Using

$$|U| \le \Delta - 1,$$

we obtain the purely structural quantity

$$\boxed{B_H - \mu_H(Q)(\Delta - 1).}$$

All of these quantities are finite.

The actual “infinity” appears only in the final step:

$$\boxed{\mathcal{E}_H(N) = \sum_{j \le N} e_{H,j}\,.}$$

If

$$\mathcal{E}_H(N)$$

is unbounded, then the cumulative number of genuine relations is also unbounded.

Therefore

infinity is not a one-step operation on an infinite object; infinity is the accumulation of finite relation certificates without an

This gives infinite relations a structure that is fully computable, provable, and transportable in batches.

It also locates the resource-lifetime theorem exactly.

A historical atom u has finite absorption lifetime:

$$\rho(u) < \infty.$$

This means that old resources cannot bear absorption responsibility forever.

New historical atoms, however, continue to appear.

Thus the global comparison is realized on one cumulative ledger: the absorption capacity of new resources is constrained by reuse degree and finite lifetime, while new relation supply continues to enter. For Twin, the unit-difference pattern has $\mu_H = 1$; the supply–absorption comparison established above yields cumulative supply strictly exceeding cumulative absorption, hence an unbounded cumulative net potential and the closure of infinite recurrence.

This global ledger is composed of the complete relation-accounting chain

$$B_H, \quad U, \quad \mu_H, \quad C_H, \quad e_H, \quad \mathcal{E}_H.$$

For Twin,

$$H = \{0,2\}, \qquad \mu_H = 1.$$

For a general prime tuple,

$$\mu_H$$

records the resource reuse degree permitted by the pattern.

For other relation orbits, the finite certification tool may change, while the infinite-quantifier interface remains unchanged.

Thus the most important unification in this part is not

> *every infinite problem uses the same sieve,*

but rather

every infinite problem must make explicit how finite facts accumulate into an unbounded or universal criterion exactly matching its own quantifier.

This also explains why

INF_RECURRENCE, UNIVERSAL_FIBER, WINDOW_UNIVERSAL, ARBITRARILY_LONG,

cannot be collapsed into one generic notion of "infinity."

They share the fixed spectrum, relational syntax, finite horizons, and certificate language;

they use different global closers at the final step.

Thus the first four parts together with this part form the complete chain

fixed identity
⇓
relation reconstruction and derived geometry
⇓
relation slices and relational normal forms of quantifiers
⇓
CRT survival tree and first local elimination
⇓
genuine death and relation alphabet
⇓
actual relation supply B_H
⇓
absorption capacity and net relation potential
⇓
cumulative certificate
⇓
global quantifier closure of the target proposition.

Finite certificates are thereby connected to the formal proofs of classical propositions; for Twin, this part has completed the global closure from finite square horizons to infinite recurrence.

Part VI will complete, proposition by proposition, the two-way equivalence between each classical proposition and its relational normal form, directly invoking the certificate theorems already established here.

Thus

classical proposition ↔ relational normal form ← finite certificate ← finite control.

Every layer has its own objects; every layer can be independently verified; and the entire proof chain passes continuously from one layer to the next.

The final summary sentence of this part is therefore

finiteness determines the identity of each individual relation; accumulation determines whether an infinite relation has an en

and

the standard for an infinite relation is not “looking far enough,”
but an unbounded structure in the accumulation of finite certificates.

Part VI Formal Proofs of Classical Propositions, the Complete Family of Relation Slices, and Method Boundaries

— From Lossless Compilation of Classical Propositions to Convolution Histories, Classical Methods, and Capability Boundaries

Central Theme of This Part

The first five parts have established:

fixed identity → relation reconstruction → finite terminality → relation slices → finite control → infinite quantifiers.

At this point, Unified Prime Relations Theory possesses a complete foundational language.

The task of this part is not to introduce another relation matrix, but to return the classical prime problems, one by one, to the matrix already constructed and to show

how the same fixed identity spectrum supports fixed sums, fixed differences, windows, affine relations, polynomial relation

At the same time, this part systematically locates the classical sieve and analytic methods.

They neither replace one another nor compete with the unified matrix.

They read different layers of the matrix:

multiplicative occupancy | shallow multiplicative depth | average distribution across moduli | fixed-sum frequency | highe

Therefore

what is unified is the object, the relational language, and the proof interface;the method may vary with the relation.

This part further upgrades “relation slices” into rigorous formal proofs. Each classical proposition is taken through the sequence

original proposition → variable normalization → position map → fixed-identity predicate → relation slice → relational norm

Hence

the classical problems are not merely “analogized” to the unified language; they are proved, item by item, to be rigorously e

Abstract

This part systematically develops the carrying capacity of the unified prime relation matrix.

Define the general relation matrix

$$\mathfrak{M} = (\boldsymbol{I}, \mathcal{A}, \Gamma, \{\mathcal{V}_t\}, \mathsf{Read}),$$

where

- $\boldsymbol{I}$ is the fixed arithmetic identity spectrum;
- $\mathcal{A}$ is the atomic layer;
- Γ is the persistent relation field;
- $\{\mathcal{V}_t\}$ is the family of finite terminal horizons;
- Read is the relation readout.

A concrete relation can be written as a quadruple

$$\boxed{\mathfrak{P} = (\Gamma_F, B, \mathcal{R}, Q),}$$

where

- $\Gamma_F = F(\mathcal{U})$ is the relation orbit;
- $B \subseteq \mathbb{N}^r$ is the admissible set of multiplicative depths;
- $\mathcal{R}$ is a finite observation region, slice, or scale;
- Q is the outer quantifier or global requirement.

For any finite-order relation space

$$\mathcal{X}_r = \mathbb{O}_{\geq 3}^r,$$

define the multiplicative-depth tensor

$$\boxed{\Delta_r(x_1, \dots, x_r) = (\Omega(x_1), \dots, \Omega(x_r)),}$$

and the atomic vertex skeleton

$$\boxed{\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1) = \mathbb{P}_{\text{odd}}^r.}$$

If

$$F = (F_1, \dots, F_r): \mathcal{U} \to \mathcal{X}_r,$$

then

$$\boxed{\{t: F_1(t), \dots, F_r(t) \text{ are all odd primes}\} = F^{-1}(\mathcal{V}_r).}$$

This shows that

unified representation is not a classification by name, but a rigorous preimage structure.

This part standardizes the formal proof of a classical proposition into nine steps:

1.	classical original proposition
2.	variable normalization
3.	position map
4.	fixed-identity predicate
5.	relation slice
6.	relational normal form of the quantifiers
7.	proof of two-way equivalence
8.	finite certificate / structural criterion
9.	translation back to the classical proposition.

Two-way relations are written with

$$\Leftrightarrow,$$

whereas a finite certificate that gives only a sufficient condition enters the target proposition strictly through

$$\Rightarrow.$$

Thus “formal equivalence” and “certificate transfer” have a uniform and explicit logical strength throughout the book.

Accordingly, this part organizes six major relation geometries:

1. fixed relation slices;
2. higher-order affine relations;
3. nonlinear relation orbits;
4. sparse exponential relation orbits;
5. interval and metric relations;
6. statistical and spectral relations.

It then compiles the classical problems, one by one, into the unified language:

- Goldbach;
- Lemoine / Lévy;
- weak Goldbach;
- Chen;
- Twin;
- Polignac;
- Sophie Germain / safe primes;
- the Hardy–Littlewood prime k-tuple conjecture;
- Green–Tao;
- Dirichlet;
- the prime number theorem and the prime number theorem in arithmetic progressions;
- Bertrand;
- Legendre;
- Oppermann;
- Brocard;
- Andrica;
- Firoozbakht;
- $n^2 + 1$;
- Bunyakovsky;
- Schinzel H;
- Mersenne;
- Fermat;
- Waring–Goldbach;
- Grimm;
- Erdős–Woods;
- the global spectral interface of RH / GRH.

These problems share the fixed identity spectrum while using different position maps, local predicates, and quantifiers.

This part also systematically studies the transfer of information from higher-order relations to lower-order relations. If

$$\pi: \mathcal{X}_r \to \mathcal{X}_s, \qquad s < r,$$

maps higher-order relation instances losslessly to lower-order relation witnesses, then higher-order structure can deliver genuine information to lower-order structure.

Typical examples include

$$p_j + p_{r-1-j} = p_0 + p_{r-1},$$

namely the mirror folding of a prime arithmetic progression into a fixed-sum relation; and the fixed-sum witness-field identity established in Part III,

$$\sum_m \left| \bigcap_{h \in H} W_K(m+h) \right| = K\, T_K(H).$$

But the ability of a higher-order structure to produce lower-order witnesses does not mean that an arbitrary higher-order average statement implies a lower-order universal proposition.

This part therefore fixes the methodological discipline

higher-order structure may add redundancy;
a lower-order proposition must still recover a lower-order witness.

Classical methods are relocated according to the information layer they preserve:

- the sieve of Eratosthenes: multiplicative occupancy;
- Brun's sieve: multi-endpoint rough relations;
- Selberg's sieve: optimization of local exclusion weights;
- combinatorial and linear sieves: shallow multiplicative depth;
- the large sieve: multi-modulus average control;
- Bombieri–Vinogradov: average distribution over moduli;
- the Hardy–Littlewood circle method: fixed-sum frequencies;
- Chen's method: relaxation of target depth under a fixed relation;
- GPY / Maynard–Tao: compression of fixed differences into a bounded band;
- Green–Tao: higher-order affine structure;
- L-function methods: global spectrum;
- Pocklington, Lucas, APR, and ECPP: finite terminal certificates.

At the analytic level, this part retains the Primorial coarse–fine decomposition

$$P = M_W + E_W,$$

Fourier-support separation

$$\operatorname{supp}\widehat{M}_W \cap \operatorname{supp}\widehat{E}_W = \varnothing,$$

and

$$M_W * E_W = 0,$$

so that

$$P^{*r} = M_W^{*r} + E_W^{*r}.$$

Define the Primorial shell

$$\Delta_j P = (\Pi_j - \Pi_{j-1})P,$$

the transverse innovation energy $\mathcal{T}_j$, and obtain

$$\boxed{\mathcal{H}_J - \mathcal{H}_k = \sum_{j=k+1}^{J} \frac{\mathcal{T}_j}{A_j}.}$$

Finite local-factor transport is written as

$$R_{r,j}(N) = \sigma_{r,p_j}(N) R_{r,j-1}(N) + \varepsilon_{r,j}(N),$$

whose iteration gives the finite Duhamel formula

$$\boxed{R_{r,J}(N) = R_{r,k}(N) \prod_{\ell=k+1}^{J} \sigma_{r,p_\ell}(N) + \sum_{m=k+1}^{J} \varepsilon_{r,m}(N) \prod_{\ell=m+1}^{J} \sigma_{r,p_\ell}(N).}$$

These analytic structures state precisely

$$\boxed{\text{local periodic skeleton} + \text{global transverse remainder.}}$$

This part also retains multiplicative-depth parity

$$\lambda(n) = (-1)^{\Omega(n)},$$

and explains that a positive sieve records only “whether a point survives,” not “whether it survives at odd or even multiplicative depth.”

When only depths 1 and 2 remain after deep sieving,

$$\boxed{\begin{pmatrix} P_z \\ Q_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} R_z \\ L_z \end{pmatrix}.}$$

The parity-resolved decomposition for fixed sums is

$$\boxed{4\mathrm{G}_z(N) = (R_z * R_z)(N) - 2(R_z * L_z)(N) + (L_z * L_z)(N).}$$

These formulas explicitly display the information layers on which traditional methods most naturally gain strength, and they also show why lower-order relations cannot be replaced directly by averages, positive energies, or single-scale uniformity.

Finally, this part defines generative complexity by

$$\boxed{\mathsf{C}_{\mathrm{gen}} = (\mathsf{C}_{\mathrm{layer}}, \mathsf{C}_{\mathrm{horizon}}, \mathsf{C}_{\mathrm{relation}}, \mathsf{C}_{\mathrm{certificate}}).}$$

This makes it possible to compare different problems structurally by

- atomic-layer cost;
- horizon cost;
- relation cost;
- certificate cost.

The part thereby completes the connection from the unified relation matrix to the full family of classical problems, the full family of classical methods, and the boundaries of their capabilities.

Keywords: classical relation slices; unified matrix; relation orbit; multiplicative depth; Goldbach; Twin; Green–Tao; sieve methods; circle method; parity; coarse–fine; higher-order relations; lower-order witnesses; generative complexity

Part I The Carrying Form of the Unified Matrix

Chapter 1 The Relation Matrix Precedes Named Problems

A concrete prime problem usually comes with a historical name.

But the name is not the underlying mathematical object.

What actually determines the structure of a problem is

which positions it reads + which identities it requires + which region it reads in + which quantifier it imposes.

Therefore

the relation matrix exists structurally before the concrete problem.

(1.1)

Here “before” denotes structural order, not chronological order.

A concrete problem does not recreate primes, nor does it recreate all basic relations among integers.

It specifies, from the matrix,

- which relation objects;
- which multiplicative identity;
- which finite or moving region;
- which global requirement.

Chapter 2 The Five Layers of the Unified Matrix

Define

$$\boxed{\mathfrak{M} = (\boldsymbol{I}, \mathcal{A}, \Gamma, \{\mathcal{V}_t\}, \mathsf{Read}).} \tag{2.1}$$

2.1 Fixed-identity layer

$$\boldsymbol{I}(n)$$

stores

$$\mathfrak{I}(n), \quad \alpha(n), \quad P^{-}(n), \quad \Omega(n), \quad \mathrm{PF}(n), \ldots$$

and answers

$$\boxed{\text{What is this integer?}}$$

2.2 Atomic layer

$$\mathcal{A}$$

records the irreducible atoms in the odd-integer multiplicative structure.

It answers

> Which positions have multiplicatively irreducible identity?

2.3 Relation layer

$$\Gamma$$

stores relations formed after atomic birth and inherited permanently thereafter.

It answers

> How do these objects stand in relation to one another?

2.4 Horizon layer

$$\{\mathcal{V}_t\}$$

records when a finite relation has received terminal certification.

It answers

> When can a finite relation no longer be rewritten by future history?

2.5 Readout layer

$$\mathsf{Read}$$

specifies

- sums;
- differences;
- windows;
- multi-endpoint patterns;
- polynomials;
- exponential orbits;
- factor-identity blocks;
- counts;
- densities;
- correlations;
- spectra.

Therefore

a new relation problem usually does not require a new atomic relation matrix; it requires only a (2.2)

Chapter 3 Arbitrary Finite-Order Relation Spaces

Define

$$\boxed{\mathcal{X}_r = \mathbb{O}_{\geq 3}^{r}.} \tag{3.1}$$

Here

$$r = 1,2,3,\ldots$$

denotes the order of the relation.

Let the odd primes be ordered by

$$p_1 < p_2 < \cdots,$$

and define the one-dimensional prime-gap cell

$$\boxed{I_n = \{p_n, p_n + 2, \ldots, p_{n+1}\}.} \tag{3.2}$$

For a multi-index

$$\boldsymbol{i} = (i_1, \ldots, i_r),$$

define

$$\boxed{\mathcal{C}_{\boldsymbol{i}} = I_{i_1} \times \cdots \times I_{i_r}.} \tag{3.3}$$

Then

$$\boxed{\mathcal{X}_r = \bigcup_{\boldsymbol{i}} \mathcal{C}_{\boldsymbol{i}}.} \tag{3.4}$$

The two-dimensional sum–difference relation plane is merely the smallest nontrivial instance $r = 2$.

Chapter 4 Multiplicative-Depth Tensor and Atomic Vertex Skeleton

Define

$$\boxed{\Delta_r(x_1, \dots, x_r) = (\Omega(x_1), \dots, \Omega(x_r)).} \tag{4.1}$$

Define

$$\boxed{\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1).} \tag{4.2}$$

Then

$$\boxed{\mathcal{V}_r = \mathbb{P}^r_{\text{odd}\cdot}} \tag{4.3}$$

Hence

every finite-order simultaneous-primality relation is an atomic-vertex relation in the same multiplicative-depth field.

(4.4)

If shallow almost-primes are allowed, one need only enlarge the target set from

$$\{(1, \dots, 1)\}$$

to

$$B \subseteq \mathbb{N}^r.$$

The original relation orbit need not change.

Chapter 5 Vertices, Edges, Faces, and Shallow Multiplicative Depth

Within a cell

$$C_i,$$

geometric position	multiplicative depth
atomic vertex	$(1, \dots, 1)$
interior of an edge	at least one component > 1
interior of a two-dimensional face	at least two components > 1
⋮	⋮

Therefore

$$\boxed{\text{prime} \quad \text{and} \quad \text{almost-prime}}$$

are not two geometries.

They are simply different layers of the same multiplicative-depth space.

Chapter 6 Relation Orbits

Let the parameter space be

$$\mathcal{U}.$$

Given

$$\boxed{F = (F_1, \dots, F_r): \mathcal{U} \to \mathcal{X}_r,} \tag{6.1}$$

define

$$\boxed{\Gamma_F = F(\mathcal{U}).} \tag{6.2}$$

and call it the relation orbit.

A relation orbit may be

- a fixed-sum slice;
- a fixed-difference slice;
- a higher-order affine line;
- a multiparameter affine plane;
- a polynomial curve;
- a sparse exponential orbit;
- a family of intervals;
- an index relation;
- a factor-identity block;
- a statistical or spectral readout.

What is truly unified is

all relation orbits enter the same fixed identity spectrum and multiplicative-depth field.

(6.3)

Chapter 7 Atomic-Vertex Preimage Theorem

Theorem 7.1

For

$$F = (F_1, \dots, F_r),$$

we have

$$\boxed{\{t \in \mathcal{U}: F_1(t), \dots, F_r(t) \text{ are all odd primes}\} = F^{-1}(\mathcal{V}_r).} \tag{7.1}$$

Proof

Since

$$\mathcal{V}_r = \Delta_r^{-1}(1, \dots, 1),$$

we have

$$F(t) \in \mathcal{V}_r$$

if and only if

$$\Omega(F_j(t)) = 1$$

for every j.

This is exactly the statement that all $F_j(t)$ are odd primes. This proves the theorem.

More generally,

$$\boxed{\{t: \Delta_r(F(t)) \in B\} = F^{-1}(\Delta_r^{-1}(B)).} \tag{7.2}$$

Therefore

unified representation is not a linguistic analogy, but a rigorous preimage structure.

(7.3)

Chapter 8 Unified Quadruple Representation of a Relation

Define

$$\boxed{\mathfrak{P} = (\Gamma_F, B, \mathcal{R}, Q).} \tag{8.1}$$

8.1 Relation orbit Γ_F

It answers

Which objects are forming the relation?

8.2 Target depth B

It answers

> Which multiplicative identities are admissible?

8.3 Observation region $\mathcal{R}$

It answers

> Within which finite region, slice, scale, or window is the relation read?

8.4 Global requirement Q

It answers

> Under which quantifier do these relations hold?

For example,

$$\exists, \quad \forall, \quad \exists^{\infty}, \quad \forall k \exists \cdots,$$

as well as count growth, density, or spectral properties.

Therefore

$$\boxed{\text{many superficially different problems differ only in one or more components of } (\Gamma_F, B, \mathcal{R}, Q).} \tag{8.2}$$

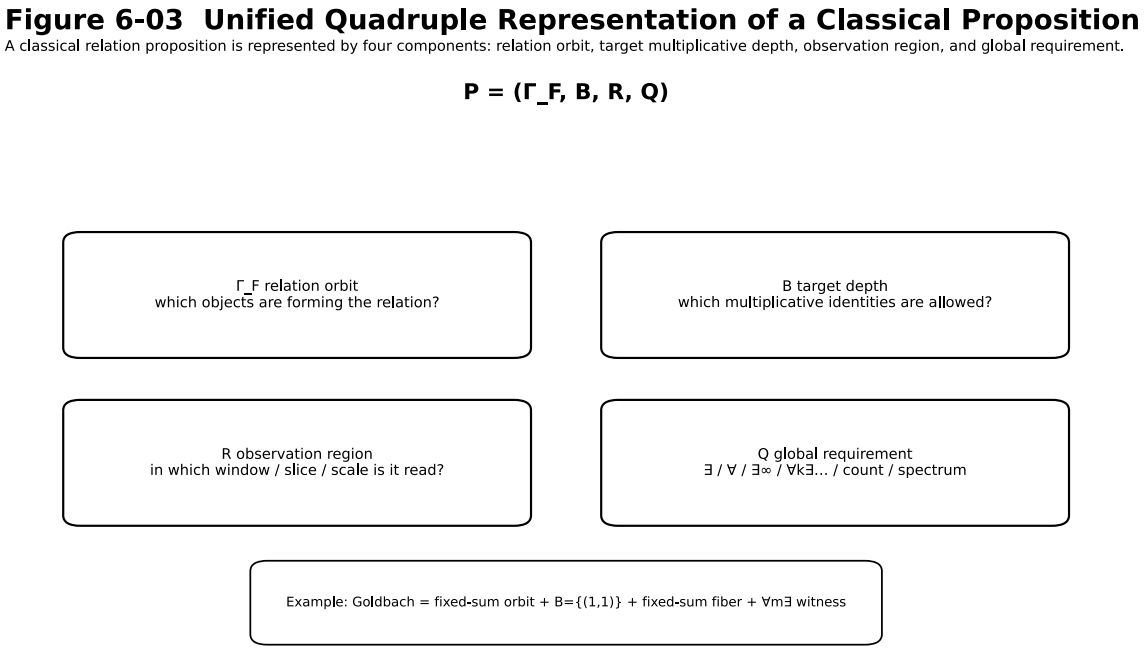


Figure 6-03. Unified Quadruple Representation of a Classical Proposition. A relation proposition is represented uniformly by $\mathfrak{P} = (\Gamma_F, B, \mathcal{R}, Q)$: Γ_F specifies the relation orbit, B the allowed multiplicative-depth set, $\mathcal{R}$ the finite region, slice, scale, or window, and Q the global requirement such as existence, universality, infinite recurrence, arbitrary length, count growth, density, or spectrum. Superficially different problems often change only one or several of these components.

Chapter 9 Terminal Certification of Finite Relations

Let

$$\mathcal{U}_0 \subset \mathcal{U}$$

be a finite parameter set.

Assume

$$3 \le F_j(t) \le H$$

for every

$$t \in \mathcal{U}_0$$

and every j.

Let

$$p^\star = \max\{p \in \mathbb{P} : p \le \sqrt{H}\}.$$

Theorem 9.1 Terminal Certification Theorem for Finite Relations

All prime/composite identities of the endpoints in the finite region $\mathcal{U}_0$ are completely determined by the finite atomic history

$$q \le p^\star.$$

Therefore

$$\boxed{\mathsf{Read}_{F,B}(\mathcal{U}_0)}$$

has a finite, complete, zero-error terminal certification.

Proof

If

$$F_j(t) \le H$$

is composite, then it has a prime divisor not exceeding its own square root:

$$q \le \sqrt{F_j(t)} \le \sqrt{H}.$$

Thus its composite identity must be detected within the finite history.

Conversely, if an integer greater than 1 has no prime divisor not exceeding its square root, then it is prime.

Since the total number of endpoints is finite, all relation identities are terminally determined together. This proves the theorem.

Chapter 10 How Finite Worlds Compose a Global Relation

Take increasing finite regions

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots,$$

such that

$$\bigcup_t \mathcal{U}_t = \mathcal{U}.$$

Each $\mathcal{U}_t$ is a complete finite relation world.

The global problem may require

$$N_t > 0 \quad \forall t,$$

or

$$N_t > 0 \quad \text{for infinitely many } t,$$

or

$$N_t \to \infty,$$

or

$$N_t \sim M(t).$$

Therefore

finite certification determines "what this layer is";

the quantifier structure determines "what all finite layers together are."

(10.1)

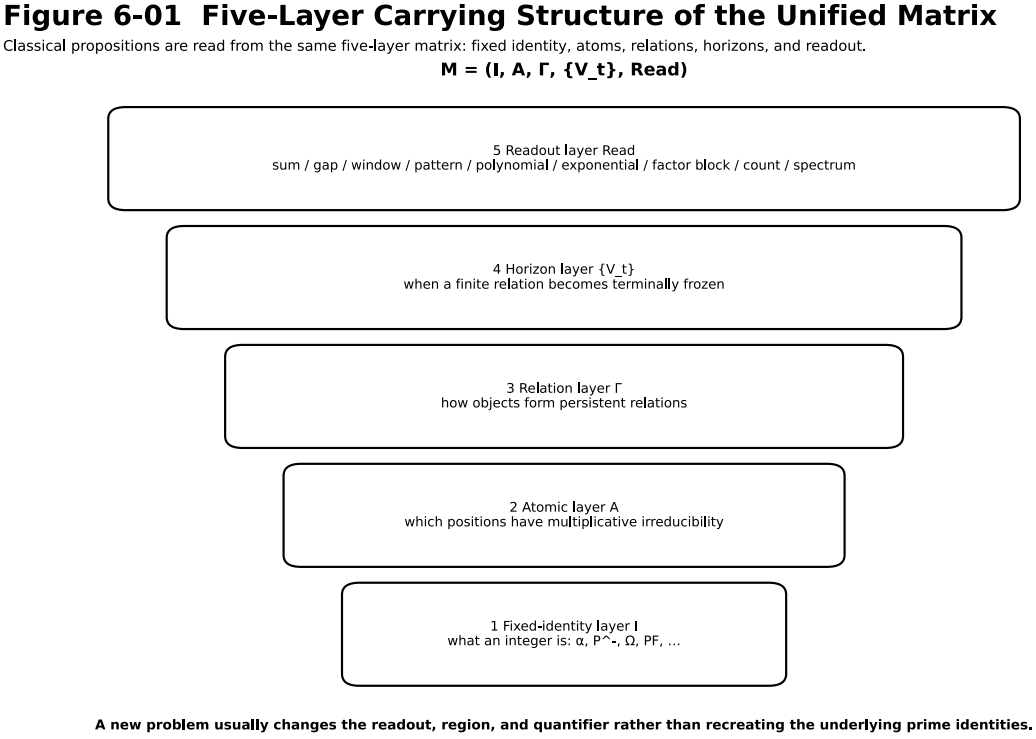


Figure 6-01. Five-Layer Carrying Structure of the Unified Matrix. The unified matrix $\mathfrak{M} = (\boldsymbol{I}, \mathcal{A}, \Gamma, \{\mathcal{V}_t\}, \mathsf{Read})$ has five layers: fixed identity says what each integer is; the atomic layer records multiplicatively irreducible positions; the relation layer preserves persistent relations among objects; the horizon layer records when a finite relation becomes terminally frozen; and the readout layer selects sums, gaps, windows, patterns, orbits, factor blocks, counts, spectra, and related observations.

Part II Six Geometries of Relations

Chapter 11 Fixed Relation Slices

The simplest relation geometry is

$$\boxed{L(x_1, \dots, x_r) = c.} \tag{11.1}$$

It selects a fixed section in the higher-dimensional relation space.

Typical examples include

- fixed sums;
- fixed differences;
- fixed linear combinations;
- fixed sums of powers in Waring–Goldbach.

Fixed sum and fixed difference are the two most basic directions among binary fixed slices.

Chapter 12 Higher-Order Affine Relations

The general form is

$$\boxed{F(t) = At + b.} \tag{12.1}$$

With a one-dimensional parameter it is an affine line;

with multidimensional parameters it is an affine relation plane.

Typical examples include

- prime k-tuples;
- Sophie Germain primes;
- Green–Tao arithmetic progressions;
- primes on congruence sublattices.

Chapter 13 Nonlinear Relation Orbits

The general form is

$$\boxed{F(n) = (f_1(n), \dots, f_r(n)),} \tag{13.1}$$

where the f_j are polynomials or other nonlinear functions.

Typical examples include

- $n^2 + 1$;
- Bunyakovsky;
- Schinzel H;
- power relations in Waring–Goldbach.

What changes is the geometry of the orbit, not the prime-identity layer.

Chapter 14 Sparse Exponential Relation Orbits

Typical forms are

$$2^p - 1,$$

$$2^{2^n} + 1.$$

These orbits are extremely sparse on the integer axis.

Ordinary interval density is not their most natural parametrization.

But the fixed spectrum still reads the identities at these sparse positions directly.

Therefore

an orbit may be extremely sparse; the matrix itself need not change.

(14.1)

Chapter 15 Interval and Metric Relations

This class does not fix a single orbit, but specifies a moving region

$$\mathcal{R}_t.$$

It reads

- whether a prime exists;
- how many primes exist;
- the prime gap;
- window length;
- scale variation.

Bertrand, Legendre, Oppermann, Brocard, and Andrica all belong to this broad class.

Chapter 16 Statistical and Spectral Relations

This class reads, from the entire atomic field,

- counts;
- densities;
- correlations;
- Fourier transforms;
- Dirichlet series;
- L-functions;
- global oscillation.

The prime number theorem, the prime number theorem in arithmetic progressions, and RH/GRH belong to this layer.

This shows that

the matrix carries not only relation existence,

but also counting, density, and spectral readouts.

(16.1)

Figure 6-02 Map of Six Relation Geometries

Part VI classifies classical propositions into six relation geometries; different geometries may call for different tools.

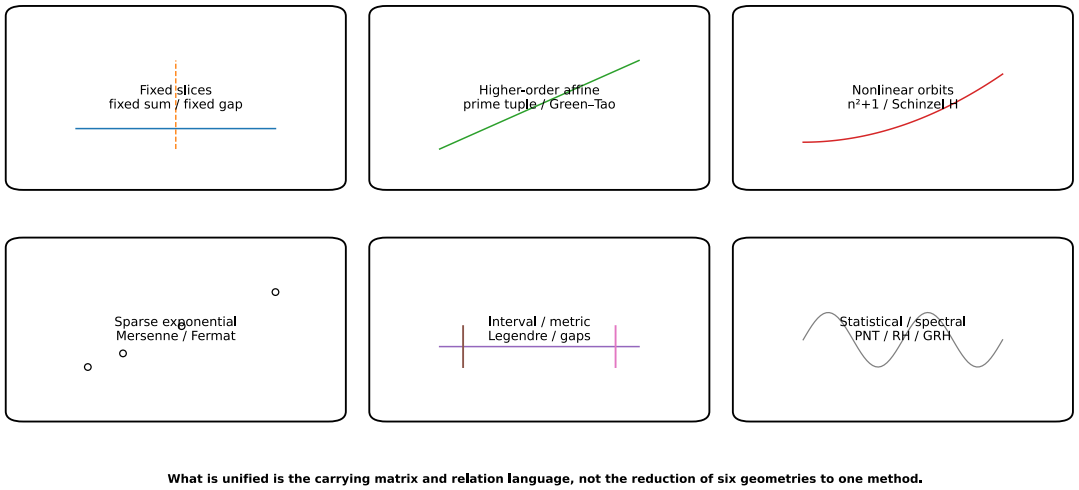


Figure 6-02. Map of Six Relation Geometries. Part VI divides the geometry of classical prime relations into six classes: fixed slices, higher-order affine relations, nonlinear orbits, sparse exponential orbits, interval/metric relations, and statistical/spectral relations. Goldbach/Twin, prime tuples/Green–Tao, n^2+1/Schinzel H, Mersenne/Fermat, Legendre/prime gaps, and PNT/RH/GRH occupy different geometries, so unification at the object layer does not imply one universal tool.

Part III Formal Proofs of Classical Propositions and the Complete Family of Relation Slices

Formal-Proof Convention for This Part

Given a classical proposition $\mathcal{P}$, write its unified relational form as

$$\mathcal{Q}_{\Theta,x}[\mathcal{B}_{\Theta}(\boldsymbol{I}(f_1(x;\Theta)),\dots,\boldsymbol{I}(f_r(x;\Theta)))].$$

The requirement for formal proof is

$$\boxed{\mathcal{P} \Leftrightarrow \mathcal{Q}_{\Theta,x}[\mathcal{B}_{\Theta}(\boldsymbol{I}(f_1),\dots,\boldsymbol{I}(f_r))].}$$

If Part V has already established a certificate C_{Θ}, then we continue with

$$\boxed{C_{\Theta} \Rightarrow \mathcal{P},}$$

or write an equivalence when a genuine two-way relation is available.

Thus

formal proof preserves the original proposition; the certificate enters the original proposition; both share the same relation o

Chapter 17 Goldbach: Universal Fibers of Fixed Sum

17.A Formal Equivalence Theorem for Goldbach

For an even integer $N \geq 6$, write $N = 2m$. Express odd-prime endpoints as

$$p = 2a + 1, \qquad q = 2b + 1, \qquad a, b \in A.$$

Then

$$p + q = N \Leftrightarrow a + b + 1 = m. \tag{17.F1}$$

Therefore:

Theorem 17.1 Formal Equivalence for Goldbach

For every $N = 2m \geq 6$, the following propositions are equivalent:

$$\exists p, q \in \mathbb{P}_{\text{odd}} : p + q = N,$$

$$\exists a, b \in A : a \boxplus b = m,$$

$$\Gamma_{\infty}^{(\sigma=m)} \neq \varnothing,$$

and

$$G(m) > 0.$$

Proof. Odd primes and atomic coordinates are in bijection through $p = 2a + 1$; equation $(17.F1)$ gives the two-way equivalence of the sum condition; by definition, $\Gamma_{\infty}^{(\sigma=m)}$ is the set of all atomic pairs satisfying $a \boxplus b = m$, and $G(m)$ is the cardinality of this fiber. This proves the theorem.

Since the least even base case

$$4 = 2 + 2$$

already holds, the entire nontrivial content of strong Goldbach begins at $N = 6$. Thus the strong Goldbach proposition is equivalent to

$$\boxed{\forall m \geq 3, \quad G(m) > 0,} \tag{17.F2}$$

equivalent to

$$\boxed{\sigma(\Gamma_{\infty}) = \mathbb{Z}_{\geq 3},} \tag{17.F3}$$

and also equivalent to

$$\boxed{A \boxplus A = \mathbb{Z}_{\geq 3}.} \tag{17.F4}$$

The natural terminal segments of Part IV further give

$$\boxed{\forall m \geq 3,\ G(m) > 0 \Leftrightarrow \forall k, \quad J_k \subseteq A_k \boxplus A_k.} \tag{17.F5}$$

Part IV has already completed this universal chain through relation completeness, placement in natural terminal segments, and terminal saturation. Thus (17.F2)–(17.F5) form the already closed main theorem chain for Goldbach in this book; the present part only compiles it losslessly into the unified relational normal form and does not repeat the main proof.

The universal-fiber certificate of Part V can also enter directly through the unified certificate language:

$$G(m) \geq L(m) > 0 \quad \forall m \Rightarrow \text{Goldbach.} \tag{17.F6}$$

Given an even integer

$$N,$$

take the position maps

$$f_1(p;N) = p, \qquad f_2(p;N) = N - p.$$

The local predicate is

$$\boxed{R_N(p) = \alpha(p)\alpha(N-p).} \tag{17.1}$$

The quantifier is

$$\boxed{\forall N \in 2\mathbb{Z}_{\geq 4}, \quad \exists p: R_N(p) = 1.} \tag{17.2}$$

Unified type:

$$\boxed{\text{UNIVERSAL_FIBER.}}$$

In atomic coordinates,

$$a + b + 1 = m.$$

A fixed sum has a natural terminal segment, so it belongs to

$$\boxed{\text{the terminal-saturation relation-sieve type.}}$$

Its complete object was established in Part IV.

Chapter 18 Lemoine / Lévy: Weighted Reflection Fibers

18.A Formal Equivalence Theorem for Lemoine / Lévy

Let $N \geq 7$ be odd. The classical relation is

$$N = p + 2q, \qquad p, q \in \mathbb{P}.$$

Set

$$q = \frac{N-p}{2}.$$

Then

$$N = p + 2q \Leftrightarrow q = \frac{N-p}{2}.$$

Theorem 18.1 Formal Equivalence for Weighted Reflection

$$\exists p, q \in \mathbb{P}: N = p + 2q$$

if and only if there exists p such that

$$\alpha(p)\alpha\left(\frac{N-p}{2}\right) = 1,$$

with $N - p > 0$ and $2 \mid N - p$.

Proof. Both directions follow directly from $q = (N-p)/2$. This proves the theorem.

Define the fiber count

$$L(N) = \sum_{p} \alpha\,(p)\alpha\left(\frac{N-p}{2}\right)\mathbf{1}_{\{N-p>0,\ 2|N-p\}}.$$

Then the universal proposition is rigorously equivalent to

$$\boxed{L(N) > 0 \quad \text{for every odd } N \geq 7.} \tag{18.F1}$$

Given odd N, require

$$N = p + 2q.$$

This can be written as

$$f_1(p; N) = p,$$

$$f_2(p; N) = \frac{N-p}{2}.$$

The local predicate is

$$\boxed{L_N(p) = \alpha(p)\alpha\left(\frac{N-p}{2}\right),} \tag{18.1}$$

with the additional requirement that

$$N - p$$

be a positive even integer.

It shares with Goldbach the type

UNIVERSAL_FIBER,

but the relation map is a weighted reflection rather than an ordinary reflection.

Chapter 19 Weak Goldbach: Raising the Relation Order to Three

19.A Ternary Formal Proof of Weak Goldbach

For odd $N \geq 7$, define

$$R_N(p,q) = \alpha(p)\alpha(q)\alpha(N-p-q).$$

Theorem 19.1 Formal Equivalence for a Ternary Fixed Sum

$$\exists p_1, p_2, p_3 \in \mathbb{P}: p_1 + p_2 + p_3 = N$$

if and only if there exist p, q such that

$$R_N(p,q) = 1.$$

Proof. In one direction take $p = p_1, q = p_2$; in the other set $p_3 = N - p - q$. This proves the theorem.

Thus the universal normal form of weak Goldbach is

$$\boxed{\forall N \in 2\mathbb{Z}+1,\ N \geq 7,\quad \sum_{p,q} R_N(p,q) > 0.} \tag{19.F1}$$

The ternary fixed sum is

$$x + y + z = N.$$

The relation space is

$$\mathcal{X}_3.$$

The target depth is

$$\boxed{\Delta_3(x,y,z) = (1,1,1).} \tag{19.1}$$

This shows that

the order of a prime additive relation corresponds directly to the dimension of the multiplicative-depth tensor.

(19.2)

A ternary fixed sum has additional convolutional redundancy; this is an important structural difference from the binary fixed sum.

Chapter 20 Chen: Same Relation Geometry, Relaxed Target Depth

20.A Formal Proof of a Chen-Type Proposition

Define the shallow multiplicative-depth indicator

$$\chi_{\leq 2}(n) = \mathbf{1}_{\{n>1,\ \Omega(n)\leq 2\}}.$$

For even N, define

$$C_N(p) = \alpha(p)\chi_{\leq 2}(N-p).$$

Theorem 20.1 Formal Equivalence of the Chen Relation

"N can be written as the sum of a prime and an integer of multiplicative depth at most 2" if and only if

$$\exists p: C_N(p) = 1.$$

Proof. The condition on the second endpoint is exactly $\Omega(N-p) \leq 2$, so the two formulations are identical. This proves the theorem.

Thus Chen and Goldbach share the relation geometry

$$x + y = N,$$

while the target depth is enlarged from

$$(1,1)$$

to

$$(1, \leq 2).$$

The binary fixed sum remains

$$x + y = N.$$

Goldbach requires

$$\Delta_2 = (1,1).$$

A Chen-type relation permits

$$\boxed{\Delta_2 \in \{(1,1),(1,2)\}} \tag{20.1}$$

or the symmetric version.

Therefore

what Chen changes is not the relation, but the admissible multiplicative depth. (20.2)

This gives a very clear place for almost-primes in the unified matrix.

Chapter 21 Twin: Infinite Recurrence of Unit Fixed Difference

21.A Formal Equivalence and Certificate Chain for Twin

Write

$$p = 2a + 1.$$

Then

$$p + 2 = 2(a + 1) + 1,$$

so

$$p, p + 2 \in \mathbb{P} \Leftrightarrow a, a + 1 \in A.$$

Theorem 21.1 Formal Equivalence for Twin

The following propositions are equivalent:

$$\exists^{\infty} p: p, p + 2 \in \mathbb{P},$$

$$\exists^{\infty} a: \hat{\alpha}(a)\hat{\alpha}(a + 1) = 1,$$

$$\#\Gamma^{[1]} = \infty,$$

$$C_T(X) = \sum_{a \le X} \hat{\alpha}\,(a)\hat{\alpha}(a + 1) \quad \text{is unbounded,}$$

and

$$\forall K\ \exists j > K: T_j > 0.$$

Proof. The first three formulations follow from the atomic coordinate and the definition of the unit-difference slice; a discrete relation set is infinite if and only if its cumulative count is unbounded; and the square horizons form an unbounded sequence of mutually disjoint finite partitions, so cumulative unboundedness is equivalent to the existence of hits in arbitrarily distant horizons. This proves the theorem.

Part V gives

$$T_j \ge B_j - |U_j| \ge B_j - (\Delta_j - 1).$$

Therefore

$$\boxed{\limsup_{N \to \infty} \mathcal{E}_T^{\text{exact}}(N) = +\infty \Rightarrow \text{Twin.}} \tag{21.F1}$$

and

$$\boxed{\limsup_{N \to \infty} \mathcal{E}_T^{\text{struct}}(N) = +\infty \Rightarrow \text{Twin.}} \tag{21.F2}$$

Part V has already completed the cumulative-supply versus cumulative-absorption comparison for the unit-difference pattern and, from $\mu_H = 1$, resource half-life, and finite lifetime, obtained unbounded cumulative net potential. Hence the closure conditions in (21.F1)–(21.F2) have already been satisfied in the main proof of Twin, and infinite recurrence of unit difference follows. As with

Goldbach, the present part retains only the formal equivalence and certificate chain and does not repeat the main proof from Part V.

The position maps are

$$f_1(p) = p, \qquad f_2(p) = p + 2.$$

The local predicate is

$$\boxed{T(p) = \alpha(p)\alpha(p+2).} \tag{21.1}$$

The quantifier is

$$\boxed{\exists^{\infty} p : T(p) = 1.} \tag{21.2}$$

Unified type:

$$\boxed{\text{INF_RECURRENCE.}}$$

It belongs to

$$\boxed{\text{the persistent non-extinction relation-sieve type.}}$$

Part V has already provided its square-horizon supply–absorption interface.

Chapter 22 Polignac: The Entire Family of Fixed Differences

22.A Formal Proof for Polignac

Fix $h \geq 1$. Define

$$R_h(p) = \alpha(p)\alpha(p+2h),$$

and

$$E_h(p) = \prod_{p<n<p+2h} (1-\alpha(n)).$$

Let

$$P_h(p) = R_h(p)E_h(p).$$

Theorem 22.1 Formal Equivalence for a Fixed Consecutive Even Gap

The primes $p, p+2h$ are consecutive if and only if

$$P_h(p) = 1.$$

Proof. $R_h = 1$ is equivalent to primality of both endpoints; $E_h = 1$ is equivalent to the absence of a prime in the open interval. Together these are exactly the condition that the two primes be consecutive. This proves the theorem.

Thus Polignac's conjecture for fixed $2h$ is equivalent to

$$\boxed{\exists^{\infty} p: P_h(p) = 1.} \tag{22.F1}$$

The full Polignac conjecture is equivalent to

$$\boxed{\forall h \geq 1, \quad \exists^{\infty} p: P_h(p) = 1.} \tag{22.F2}$$

Fix the even difference

$$2h.$$

The positions are

$$p, \qquad p+2h.$$

The local predicate is

$$\boxed{R_h(p) = \alpha(p)\alpha(p+2h).} \tag{22.1}$$

If the two endpoints are required to be consecutive primes, add

$$\boxed{E_h(p) = \prod_{p<n<p+2h} (1-\alpha(n)).} \tag{22.2}$$

Thus

$$\boxed{P_h(p) = R_h(p)E_h(p).} \tag{22.3}$$

Twin corresponds to $h = 1$.

Cousin primes correspond to $h = 2$.

Sexy primes correspond to $h = 3$.

Hence

Twin is the smallest nontrivial member of the fixed-difference family.

(22.4)

Chapter 23 Sophie Germain and Safe Primes: A Slanted Relation

23.A Formal Proof for Sophie Germain / Safe Primes

Define

$$SG(p) = \alpha(p)\alpha(2p+1).$$

Theorem 23.1 Formal Equivalence for the Slanted Relation

p is a Sophie Germain prime if and only if

$$SG(p) = 1.$$

A number q is the corresponding safe prime if and only if there exists

$$p = \frac{q-1}{2}$$

with $SG(p) = 1$.

Thus infinitude is rigorously equivalent to

$$\boxed{\sum_{p \le X} SG(p) \quad \text{being unbounded.}} \tag{23.F1}$$

Define

$$\boxed{F(p) = (p, 2p+1).} \tag{23.1}$$

The local predicate is

$$\boxed{SG(p) = \alpha(p)\alpha(2p+1).} \tag{23.2}$$

A safe prime is the reverse readout of the same relation.

Therefore

reversing the direction of a relation does not require changing the prime relation matrix.

(23.3)

Chapter 24 Hardy–Littlewood Prime k-Tuples

24.A Formal Proof for the Hardy–Littlewood Prime k-Tuple

Given

$$H = \{h_1, \dots, h_r\},$$

define

$$R_H(n) = \prod_{j=1}^{r} \alpha\,(n + h_j).$$

Theorem 24.1 Formal Equivalence for a Fixed Pattern

$$R_H(n) = 1$$

if and only if

$$n + h_1, \dots, n + h_r$$

are all prime.

Therefore infinite recurrence of the pattern is equivalent to

$$\boxed{C_H(X) = \sum_{n \le X} R_H\,(n) \quad \text{being unbounded.}} \tag{24.F1}$$

Local admissibility is written as

$$\boxed{\nu_H(q) < q \quad \text{for every prime } q.} \tag{24.F2}$$

Part IV expands this into the CRT survival tree; Part V constructs $B_H, U, \mu_H, e_H, \mathcal{E}_H$. Hence

$$\boxed{\text{limsup}\mathcal{E}_H(X) = +\infty \Rightarrow C_H(X) \text{ is unbounded.}} \tag{24.F3}$$

Take a finite set of offsets

$$H = \{h_1, \dots, h_r\}.$$

Define

$$\boxed{F(n) = (n + h_1, \dots, n + h_r).} \tag{24.1}$$

The local relation is

$$\boxed{R_H(n) = \prod_{j=1}^{r} \alpha\,(n + h_j).} \tag{24.2}$$

This is a higher-dimensional affine orbit.

Complete local coverage is detected by

$$\nu_H(q) = q.$$

Thus

$$\boxed{\text{prime constellation} = \text{recurrence of a locally admissible affine orbit to the higher-dimensional a}} \tag{24.3}$$

Chapter 25 Green–Tao: Arbitrarily Long Arithmetic-Progression Relations

25.A Formal Proof for Green–Tao

Define

$$C_k(a,d) = \prod_{j=0}^{k-1} \alpha\,(a + jd), \qquad d \geq 1.$$

Theorem 25.1 Formal Equivalence for Arbitrarily Long Arithmetic Progressions

"Primes contain arbitrarily long arithmetic progressions" if and only if

$$\boxed{\forall k \geq 2,\ \exists a, d \geq 1 \colon C_k(a,d) = 1.} \tag{25.F1}$$

Proof. A product of $0/1$-valued factors equals 1 if and only if every factor equals 1. This proves the theorem.

Thus its quantifier type is

$$\text{ARBITRARILY_LONG}.$$

Define

$$\boxed{F(a,d) = (a, a+d, \ldots, a+(k-1)d).} \tag{25.1}$$

The local predicate is

$$\boxed{C_k(a,d) = \prod_{j=0}^{k-1} \alpha\,(a + jd).} \tag{25.2}$$

The quantifier is

$$\boxed{\forall k\ \exists a, d \colon C_k(a,d) = 1.} \tag{25.3}$$

Unified type:

$$\boxed{\text{ARBITRARILY_LONG}.}$$

This is a problem in which a two-parameter relation manifold intersects atomic vertex skeletons of arbitrarily high dimension.

Chapter 26 Mirror Folding from Green–Tao to a Fixed Sum

26.A Mirror-Folding Theorem

Suppose

$$p_j = p_0 + jd, \qquad 0 \le j \le r-1.$$

Then

$$p_j + p_{r-1-j} = 2p_0 + (r-1)d.$$

Hence a prime arithmetic progression of length r produces

$$\boxed{\left\lceil \frac{r}{2} \right\rceil}$$

unordered prime representations of the same fixed sum

$$N = 2p_0 + (r-1)d.$$

This is a rigorous higher-order-to-lower-order witness map, because the two concrete endpoints of every lower-order witness are preserved.

For the prime arithmetic progression

$$p_j = p_0 + jd, \qquad 0 \le j \le r-1,$$

we have

$$\boxed{p_j + p_{r-1-j} = 2p_0 + (r-1)d.} \tag{26.1}$$

Let

$$N = 2p_0 + (r-1)d.$$

The same higher-order relation outputs to the fixed-sum slice

$$\boxed{\left\lceil \frac{r}{2} \right\rceil}$$

unordered prime representations.

Therefore

higher-order arithmetic-progression relations can deliver genuine witnesses to lower-order fixed sums.

(26.2)

This is a legitimate higher-order-to-lower-order transfer because every lower-order representation retains its concrete endpoints.

Chapter 27 Dirichlet: Infinite Occupancy on Congruence Sublattices

27.A Relational Form of Dirichlet's Theorem

Fix

$$(a, q) = 1.$$

Define

$$R_{a,q}(n) = \alpha(a + qn),$$

$$C_{a,q}(X) = \sum_{0 \le n \le X} R_{a,q}\,(n).$$

Theorem 27.1 Formal Equivalence for Infinite Occupancy of a Congruence Sublattice

The arithmetic progression

$$a,\ a + q,\ a + 2q, \ldots$$

contains infinitely many primes if and only if

$$\boxed{C_{a,q}(X) \text{ is unbounded.}} \tag{27.F1}$$

Thus Dirichlet's theorem is written in relational language as

$$\boxed{(a, q) = 1 \Rightarrow C_{a,q}(X) \text{ is unbounded.}} \tag{27.F2}$$

Fix

$$a \bmod q, \qquad (a, q) = 1.$$

Define

$$\boxed{F(n) = a + qn.} \tag{27.1}$$

Dirichlet's theorem states that

> every locally admissible one-dimensional congruence sublattice is occupied infinitely often by depth-1 atoms.

(27.2)

It is the classical linear reference for "local admissibility → infinite occupancy."

Chapter 28 PNT and PNT in AP: Coarse Density

28.A Statistical Forms of PNT and PNT in AP

Since

$$\pi(x) = \sum_{n \le x} \alpha\,(n),$$

the fixed-spectrum form of the prime number theorem is simply

$$\boxed{\sum_{n \le x} \alpha\,(n) \sim \frac{x}{\log x}.} \tag{28.F1}$$

For fixed $(a, q) = 1$, define

$$\pi(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a\ (q)}} \alpha\,(n).$$

Then the PNT in arithmetic progressions is written

$$\boxed{\pi(x; q, a) \sim \frac{\mathrm{Li}(x)}{\varphi(q)}.} \tag{28.F2}$$

These propositions read asymptotic density from the fixed spectrum, rather than the nonemptiness of a single fiber.

PNT reads

$$\boxed{\pi(x) = \#\{p \le x : p \in \mathbb{P}\}.} \tag{28.1}$$

Its principal scale is

$$\boxed{\pi(x) \sim \frac{x}{\log x}.} \tag{28.2}$$

PNT in AP reads density on a fixed congruence sublattice.

Such questions ask not only

Does one exist?

but

$$\boxed{\text{At what average density do the atoms occur?}} \tag{28.3}$$

Thus they belong to statistical readout rather than finite terminal readout of relations.

Chapter 29 Bertrand: Occupancy of a Moving Interval

29.A Window-Form Proof of Bertrand's Postulate

Define

$$W_B(x) = \sum_{x<n<2x} \alpha\,(n).$$

Theorem 29.1 Formal Equivalence for Bertrand

$$\forall x > 1,\ \exists p \in \mathbb{P}: x < p < 2x$$

if and only if

$$\boxed{\forall x > 1, \quad W_B(x) \geq 1.} \tag{29.F1}$$

because $W_B(x)$ counts exactly the primes in the window. This proves the theorem.

Given $x > 1$, take the window

$$(x, 2x).$$

The relation predicate is

$$\boxed{W_B(x) = \sum_{x<n<2x} \alpha\,(n).} \tag{29.1}$$

Bertrand's postulate gives

$$\boxed{W_B(x) \geq 1.} \tag{29.2}$$

It is a classical interval instance of

$$\boxed{\text{WINDOW_UNIVERSAL.}}$$

Chapter 30 Legendre: Windows Between Consecutive Integer Squares

30.A Window-Form Proof of Legendre's Conjecture

Define

$$W_L(n) = \sum_{n^2<m<(n+1)^2} \alpha\,(m).$$

Theorem 30.1 Formal Equivalence for Legendre

$$\forall n \geq 1,\ \exists p \in \mathbb{P}: n^2 < p < (n+1)^2$$

if and only if

$$\boxed{\forall n \geq 1, \quad W_L(n) \geq 1.} \tag{30.F1}$$

Thus any uniform positive lower-bound certificate

$$W_L(n) \geq L(n) > 0$$

enters the proposition directly.

Define

$$\boxed{I_n = (n^2, (n+1)^2).} \tag{30.1}$$

The window occupancy is

$$\boxed{W_L(n) = \sum_{n^2<m<(n+1)^2} \alpha\,(m).} \tag{30.2}$$

The target form is

$$W_L(n) \geq 1 \quad \forall n.$$

This shares a square scale with the prime-square horizons of Part V, but it is not the same readout:

- a prime-square horizon is a boundary for identity certification;
- a Legendre window is a region whose occupancy is to be tested.

Therefore

> **sharing a square scale does not mean sharing the concrete relation.**

(30.3)

Chapter 31 Oppermann: Universality of Two Windows

31.A Two-Window Formal Proof of Oppermann's Conjecture

Define

$$W_{-}(n) = \sum_{n(n-1)<m<n^2} \alpha(m),$$

$$W_{+}(n) = \sum_{n^2<m<n(n+1)} \alpha(m).$$

Theorem 31.1 Formal Equivalence for Oppermann

For every $n > 1$, each of the two windows contains at least one prime if and only if

$$\boxed{W_{-}(n) \geq 1 \quad \text{and} \quad W_{+}(n) \geq 1.} \tag{31.F1}$$

Thus the universal normal form is

$$\boxed{\forall n > 1, \quad \min\{W_{-}(n), W_{+}(n)\} \geq 1.} \tag{31.F2}$$

Define

$$I_n^{-} = (n(n-1), n^2),$$

$$I_n^{+} = (n^2, n(n+1)).$$

Define

$$W_{-}(n) = \sum_{m \in I_n^{-}} \alpha(m),$$

$$W_{+}(n) = \sum_{m \in I_n^{+}} \alpha(m).$$

The local predicate is

$$\boxed{W_{-}(n) \geq 1 \quad \text{and} \quad W_{+}(n) \geq 1.} \tag{31.1}$$

Thus this is a two-window relation sharing a square boundary.

Chapter 32 Brocard: Counting Primes Between Consecutive Prime Squares

32.A Square-Window Counting Form of Brocard's Conjecture

Let p_n be the n-th prime, $n \geq 2$. Define

$$B(n) = \pi(p_{n+1}^2) - \pi(p_n^2).$$

Theorem 32.1 Formal Equivalence for Brocard

There are at least four primes between

$$p_n^2 \quad \text{and} \quad p_{n+1}^2$$

if and only if

$$\boxed{B(n) \geq 4.} \tag{32.F1}$$

Thus its universal normal form is

$$\boxed{\forall n \geq 2, \quad B(n) \geq 4.} \tag{32.F2}$$

Let

$$p_n < p_{n+1}$$

be consecutive primes.

Define

$$\boxed{B(n) = \pi(p_{n+1}^2) - \pi(p_n^2).} \tag{32.1}$$

The Brocard-type predicate is

$$\boxed{B(n) \geq 4} \tag{32.2}$$

on its standard index range.

Here the parameter itself is generated by the prime sequence.

Therefore this is

$$\boxed{\text{a prime-index-generated window} + \text{a window-counting predicate.}}$$

Chapter 33 Andrica: A Metric Relation for Prime Gaps

33.A Equivalent Metric Form of Andrica's Conjecture

The identity

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}}$$

gives:

Theorem 33.1 Formal Equivalence for Andrica

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

if and only if

$$\boxed{p_{n+1} - p_n < \sqrt{p_{n+1}} + \sqrt{p_n}.} \tag{33.F1}$$

Thus its universal normal form is

$$\boxed{\forall n \geq 1, \quad A_n < 1.} \tag{33.F2}$$

Define

$$\boxed{A_n = \sqrt{p_{n+1}} - \sqrt{p_n}.} \tag{33.1}$$

Equivalently,

$$A_n = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}}.$$

Thus it directly compares the prime gap

$$g_n = p_{n+1} - p_n$$

with the local scale

$$\sqrt{p_n}.$$

Therefore

Andrica is a local metric constraint on the prime-gap cell complex.

(33.2)

Chapter 34 Firoozbakht: Normalized Growth of Consecutive Primes

34.A Equivalent Logarithmic Form of Firoozbakht's Conjecture

Since the logarithm is strictly increasing,

$$p_{n+1}^{1/(n+1)} < p_n^{1/n}$$

if and only if

$$\boxed{\frac{\log p_{n+1}}{n+1} < \frac{\log p_n}{n}.} \tag{34.F1}$$

Thus the classical proposition can be read completely equivalently in normalized logarithmic coordinates.

The local predicate is

$$\boxed{p_{n+1}^{1/(n+1)} < p_n^{1/n}.} \tag{34.1}$$

Its equivalent logarithmic form is

$$\boxed{\frac{\log p_{n+1}}{n+1} < \frac{\log p_n}{n}.} \tag{34.2}$$

This is a relation between consecutive prime indices.

It reads not pointwise prime/composite identity, but normalized growth of the consecutive-prime sequence.

Chapter 35 $n^2 + 1$: A Quadratic Orbit

35.A Formal Proof for $n^2 + 1$

Define

$$R_2(n) = \alpha(n^2 + 1).$$

Then

$$R_2(n) = 1 \Leftrightarrow n^2 + 1 \in \mathbb{P}.$$

Therefore:

Theorem 35.1 Formal Equivalence for Infinitude on the Quadratic Orbit

"There are infinitely many n for which $n^2 + 1$ is prime" if and only if

$$\boxed{C_2(X) = \sum_{n \le X} \alpha\,(n^2 + 1) \text{ is unbounded.}} \tag{35.F1}$$

Define

$$\boxed{F(n) = n^2 + 1.} \tag{35.1}$$

The local predicate is

$$\boxed{R(n) = \alpha(n^2 + 1).} \tag{35.2}$$

If the quantifier asks for infinitely many occurrences, the proposition belongs to

INF_RECURRENCE.

This is the standard "curve readout" instance among Landau's four problems.

Chapter 36 Bunyakovsky: General Polynomial Orbits

36.A Formal Proof for Bunyakovsky

Given an integer polynomial f, define

$$R_f(n) = \alpha(f(n)).$$

Whenever $f(n) > 1$,

$$R_f(n) = 1 \Leftrightarrow f(n) \in \mathbb{P}.$$

Theorem 36.1 Formal Equivalence for Prime Values of a Polynomial

"$f(n)$ assumes prime values infinitely often" if and only if

$$\boxed{C_f(X) = \sum_{\substack{n \le X \\ f(n)>1}} \alpha\left(f(n)\right) \text{ is unbounded.}} \tag{36.F1}$$

"No fixed prime divisor" is written as

$$\boxed{\forall q \in \mathbb{P},\ \exists n: f(n) \not\equiv 0 \pmod{q}.} \tag{36.F2}$$

Given an integer-valued polynomial

$$f(n),$$

define

$$\boxed{R_f(n) = \alpha(f(n)).} \tag{36.1}$$

This may be understood as

> whether the polynomial orbit repeatedly intersects the atomic layer.

(36.2)

Linear polynomials correspond to Dirichlet-type orbits;

higher-degree polynomials retain the same fixed identity spectrum and change only the orbit geometry.

Chapter 37 Schinzel H: Polynomial Families

37.A Formal Proof for Schinzel H

Given

$$\boldsymbol{f} = (f_1, \dots, f_r),$$

define

$$R_{\boldsymbol{f}}(n) = \prod_{j=1}^{r} \alpha\,(f_j(n)).$$

Theorem 37.1 Formal Equivalence for a Polynomial Family

All $f_j(n)$ are simultaneously prime if and only if

$$R_{\boldsymbol{f}}(n) = 1.$$

Therefore "simultaneous prime values occur infinitely often" if and only if

$$\boxed{C_{\boldsymbol{f}}(X) = \sum_{n \le X} R_{\boldsymbol{f}}\,(n) \text{ is unbounded.}} \tag{37.F1}$$

Local admissibility is written

$$\boxed{\forall q \in \mathbb{P},\ \exists n: \prod_{j=1}^{r} f_j\,(n) \not\equiv 0 \pmod{q}.} \tag{37.F2}$$

Given

$$f_1, \dots, f_r,$$

define

$$\boxed{F(n) = (f_1(n), \dots, f_r(n)).} \tag{37.1}$$

The local predicate is

$$\boxed{R_{\boldsymbol{f}}(n) = \prod_{j=1}^{r} \alpha\,(f_j(n)).} \tag{37.2}$$

The complete local-coverage test asks:

> Is there a prime q such that every input residue class makes at least one $f_j(n) \equiv 0 \pmod{q}$?

Thus the structure is

$$\boxed{\text{higher-dimensional polynomial orbit} + \text{local admissibility} + \text{infinite recurrence.}} \tag{37.3}$$

Chapter 38 Mersenne: A Sparse Exponential Orbit

38.A Formal Proof for Mersenne Primes

Define

$$M_p = 2^p - 1, \qquad p \in \mathbb{P},$$

and

$$R_M(p) = \alpha(M_p).$$

Then

$$R_M(p) = 1 \Leftrightarrow M_p \text{ is a Mersenne prime.}$$

Therefore:

Theorem 38.1 Formal Equivalence for Infinitely Many Mersenne Parameters

"There are infinitely many Mersenne primes" if and only if

$$\boxed{P_M(X) = \sum_{\substack{p \le X \\ p \in \mathbb{P}}} \alpha\,(2^p - 1) \text{ is unbounded.}} \tag{38.F1}$$

Single-point Lucas–Lehmer certification belongs to finite certificates; infinitude of successful parameters belongs to

PARAMETER_INFINITE.

Given prime exponent p, define

$$\boxed{M_p = 2^p - 1.} \tag{38.1}$$

The local predicate is

$$\boxed{R_M(p) = \alpha(2^p - 1).} \tag{38.2}$$

If infinitude is the target, count

$$\boxed{C_M(X) = \sum_{\substack{p \le X \\ p \in \mathbb{P}}} R_M\,(p).} \tag{38.3}$$

This belongs to the class of sparse exponential orbits.

For an individual finite instance, Lucas–Lehmer provides a highly adapted certification tool.

Chapter 39 Fermat: A Doubly Exponential Sparse Orbit

39.A Formal Proof for the Fermat-Prime Orbit

Define

$$F_n = 2^{2^n} + 1, \qquad R_F(n) = \alpha(F_n).$$

Then

$$R_F(n) = 1 \Leftrightarrow F_n \in \mathbb{P}.$$

Thus, if the proposition asks for infinitely many Fermat primes, it is rigorously equivalent to

$$\boxed{C_F(X) = \sum_{0 \le n \le X} \alpha\left(2^{2^n} + 1\right) \text{ being unbounded.}} \tag{39.F1}$$

Define

$$\boxed{F_n = 2^{2^n} + 1.} \tag{39.1}$$

Read

$$\boxed{\alpha(F_n).}$$

This orbit is even sparser than the Mersenne orbit.

It shows that

linear, polynomial, exponential, and doubly exponential orbits can all read the same fixed identity spectrum.

(39.2)

Chapter 40 Waring–Goldbach: A Nonlinear Fixed-Sum Manifold

40.A Formal Proof of a Waring–Goldbach-Type Proposition

Fix k, r, and define

$$\mathcal{W}_{k,r}(N) = \left\{ (p_1, \dots, p_r) \in \mathbb{P}^r : \sum_{j=1}^{r} p_j^k = N \right\}.$$

Theorem 40.1 Formal Equivalence for a Nonlinear Fixed Sum

N has a representation as a sum of r k-th powers of primes if and only if

$$\boxed{\mathcal{W}_{k,r}(N) \neq \varnothing.} \tag{40.F1}$$

Thus any universal Waring–Goldbach-type proposition can be written as

$$\boxed{\forall N \in \mathcal{D}_{k,r},\ \mathcal{W}_{k,r}(N) \neq \varnothing,} \tag{40.F2}$$

where $\mathcal{D}_{k,r}$ is the corresponding admissible target domain.

The relation is

$$\boxed{p_1^k + \cdots + p_r^k = N.} \tag{40.1}$$

The matrix need not define a separate notion of "k-th-power prime."

It keeps

$$\Delta_r(p_1, \dots, p_r) = (1, \dots, 1),$$

and changes only the relation manifold

$$x_1^k + \cdots + x_r^k = N.$$

Therefore

what changes is the relation geometry, not the atomic layer.

(40.2)

Chapter 41 Grimm: Complete Factor Sets and Matching

41.A Matching-Form Proof of Grimm's Conjecture

Suppose

$$n+1, \dots, n+k$$

are all composite. Construct a bipartite graph with

- left vertices $L = \{1, \dots, k\}$;
- right vertices $R = \bigcup_i \mathrm{PF}(n+i)$;
- an edge (i, p) whenever $p \mid n+i$.

Theorem 41.1 Formal Equivalence for Grimm

The following are equivalent:

1. there exist pairwise distinct primes $p_1, \dots, p_k$ with $p_i \mid n+i$;
2. the bipartite graph above has a matching covering every left vertex.

Proof. Distinct p_i directly define such a matching; conversely, a matching covering all left vertices chooses a distinct prime divisor for every position. This proves the theorem.

By Hall's theorem, these are also equivalent to

$$\boxed{\left|\bigcup_{i\in J} \mathrm{PF}(n+i)\right| \geq |J| \quad \forall J \subseteq \{1, \dots, k\}.} \tag{41.F1}$$

Consider a consecutive block of composite integers

$$n+1, \dots, n+k.$$

At each position read

$$\mathrm{PF}(n+i).$$

Construct the bipartite graph:

- left side: positions $1, \dots, k$;
- right side: the prime factors that occur;
- connect position i to prime factor p whenever $p \mid n+i$.

The local relation asks for a matching assigning pairwise distinct prime factors to all positions.

Therefore

Grimm is not a binary prime/composite relation, but a combinatorial matching relation on complete factor sets.

(41.1)

Its outer layer is an indexwise universal predicate over blocks.

Chapter 42 Erdős–Woods: Identity-Block Separation

42.A Identity-Block Formal Proof for Erdős–Woods

Define

$$B_K(x) = (\mathrm{PF}(x), \ldots, \mathrm{PF}(x+K-1)).$$

Theorem 42.1 Formal Equivalence for Identity-Block Separation

"There exists a uniform finite K such that a block of consecutive integers is uniquely determined by its positionwise prime-factor sets" if and only if

$$\boxed{\exists K \geq 1,\ \forall x \neq y,\ B_K(x) \neq B_K(y).} \tag{42.F1}$$

Equivalently,

$$\boxed{B_K\colon \mathbb{N} \to \mathcal{P}(\mathbb{P})^K \text{ is injective.}} \tag{42.F2}$$

This is precisely

BLOCK_SEPARATION.

Define

$$\boxed{B_K(x) = (\mathrm{PF}(x), \ldots, \mathrm{PF}(x+K-1)).} \tag{42.1}$$

Compare

$$B_K(x) \quad \text{and} \quad B_K(y).$$

The global interface is

$$\boxed{\text{BLOCK_SEPARATION.}}$$

It does not ask "does the relation occur infinitely often?" but rather

> does there exist a uniform finite block length sufficient to distinguish the complete factor identities of different starting points?

(42.2)

Chapter 43 Four Different Readouts of Landau's Four Problems

43.A Unified Formal Table for Landau's Four Problems

The four classical propositions become, respectively,

$$\boxed{\text{Goldbach: } \forall m \geq 3,\ G(m) > 0,}$$

$$\boxed{\text{Twin: } C_T(X) \text{ is unbounded,}}$$

$$\boxed{\text{Legendre: } \forall n \geq 1,\ W_L(n) \geq 1,}$$

$$\boxed{n^2+1\text{: } C_2(X) = \sum_{n \leq X} \alpha\,(n^2+1) \text{ is unbounded.}}$$

Therefore

sum, difference, window, and curve propositions invoke four different quantifier closers on the same fixed spectrum.

(43.F1)

Landau's four classical problems display four genuinely different relation geometries:

Problem	Unified readout
Goldbach	fixed-sum fiber
Twin	infinite recurrence of a fixed difference
Legendre	universal occupancy of square windows
n^2+1	infinite recurrence on a quadratic orbit

Therefore

sum, difference, window, and curve readouts can all be carried by the same fixed spectrum.

(43.1)

This is deeper than merely listing the four problems together in a historical table.

What actually changes is

$$\Gamma_F, \quad \mathcal{R}, \quad Q.$$

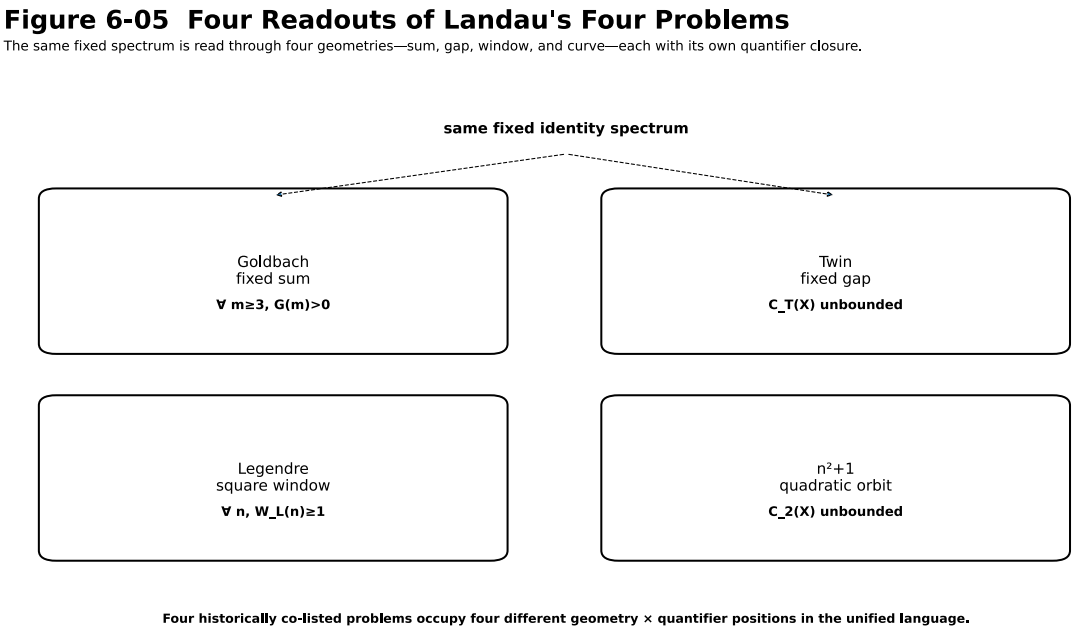


Figure 6-05. Four Readouts of Landau's Four Problems. Landau's four problems invoke four different relation geometries and quantifier closures on the same fixed spectrum: Goldbach is a universal fixed-sum fiber $\forall m \geq 3,\ G(m) > 0$; Twin is fixed-gap infinite recurrence with unbounded $C_T(X)$; Legendre is universal occupancy of square windows; and $n^2 + 1$ is infinite hitting along a quadratic orbit. What changes is the relation orbit, observation region, and outer quantifier.

Chapter 44 RH / GRH: The Global Spectral Interface

44.A Global Spectral Form of RH / GRH

Construct the von Mangoldt data $\Lambda(n)$ from the fixed arithmetic identities. For $\Re s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

The spectral statement of RH is

$$\boxed{\forall \rho \in Z_{\mathrm{nt}}(\zeta), \quad \Re\rho = \frac{1}{2}.} \tag{44.F1}$$

For the corresponding Dirichlet L-functions, GRH is written

$$\boxed{\forall \chi,\ \forall \rho \in Z_{\mathrm{nt}}(L(s,\chi)), \quad \Re\rho = \frac{1}{2}.} \tag{44.F2}$$

Thus the fixed spectrum supplies the arithmetic coefficients and Euler-product data, while the spectral proposition belongs to the global analytic-transform layer.

RH / GRH do not read a single finite relation instance.

Starting from prime data, they pass through

$$\boxed{\text{counting measure} \to \text{Dirichlet series / } L\text{-function} \to \text{analytic continuation} \to \text{zero spectrum.}} \tag{44.1}$$

Therefore

RH / GRH obtain their data source from the fixed spectrum, but belong to an additional global analytic-transform layer.

(44.2)

Finite relation sieves and global L-function spectra must not be conflated into the same operation.

Part IV Convolution Histories, Terminal Factorization, and Derived Structures

Convolution Exponential: One Parameter Encodes All Finite Relation Orders

For any function f, define the convolution exponential

$$\boxed{\mathcal{E}_u(f) = \mathrm{Exp}_*(uf) = \sum_{r=0}^{\infty} \frac{u^r}{r!} f^{*r}.} \tag{IV.1}$$

where $f^{*0} = \delta_0$.

Theorem IV.1 Homomorphism Property of the Convolution Exponential

$$\boxed{\mathcal{E}_u(f+g) = \mathcal{E}_u(f) * \mathcal{E}_u(g).} \tag{IV.2}$$

Proof. Apply the convolution binomial theorem to $(f+g)^{*r}$ and then regroup the two convolution orders. This proves the theorem.

Theorem IV.2 Evolution Equation in Relation Order

$$\boxed{\partial_u \mathcal{E}_u(f) = f * \mathcal{E}_u(f).} \tag{IV.3}$$

Therefore

one generating parameter u simultaneously encodes unary, binary, ternary, and all finite-order relations.

(IV.4)

Finite Terminal Convolution Factorization

Fix the window $[1, X]$. Let

- O_X: candidate odd positions in the window;
- P_X: genuine prime positions;

- $D_{q,X}$: the genuine death layer whose least prime factor is exactly q.

Uniqueness of genuine death gives

$$\boxed{O_X = P_X \sqcup \bigsqcup_{q\le\sqrt{X}} D_{q,X}.} \tag{IV.5}$$

Thus

$$P_X = O_X - \sum_{q\le\sqrt{X}} D_{q,X}.$$

Applying the convolution exponential,

$$\boxed{\mathcal{E}_u(P_X) = \mathcal{E}_u(O_X) * \prod_{q\le\sqrt{X}}^{*} \mathcal{E}_u\,(-D_{q,X}).} \tag{IV.6}$$

Therefore

all finite-order prime relations can be factored uniformly into the candidate relation tower and finitely many genuine-death generators.

(IV.7)

Since convolution is commutative, permuting the death factors does not alter the terminal result; hence the terminal relation tower is path-independent.

Zero Mean on Coarse–Fine Fibers

Part VII will define the coarse projection M_W and fine residual

$$E_W = f - M_W.$$

If the coarse fiber is fixed as $a + W\mathbb{Z}$, with T positions in one finite period, then

$$M_W(a + Wt) = \frac{1}{T}\sum_{u=0}^{T-1} f\,(a + Wu).$$

Therefore:

Theorem IV.3 Zero Mean on Fibers

$$\boxed{\sum_{t=0}^{T-1} E_W\,(a + Wt) = 0.} \tag{IV.8}$$

This shows that

the fine residual is not an arbitrary error term; it has exactly zero mean on every coarse fiber.

(IV.9)

Seven Derived Relation Structures

The relation layer of the entire book can be summarized by seven commonly used derived objects:

1. Relation capacity

For example,

$$\rho_H(S), \qquad \mu_H(Q)|U|.$$

2. Relation lifetime

For example, the single-resource lifetime

$$\rho(u).$$

3. Relation alphabet

$$\Sigma_{\mathcal{R}}.$$

4. Relation freezing

After finite terminality, endpoint identities are no longer changed by higher control layers.

5. Relation certificate

For example,

$$B_H > \mu_H(Q)|U| \Rightarrow T_H \geq 1.$$

6. Generative complexity

$$\mathsf{C}_{\text{gen}} = (\mathsf{C}_{\text{layer}}, \mathsf{C}_{\text{horizon}}, \mathsf{C}_{\text{relation}}, \mathsf{C}_{\text{certificate}}).$$

7. Information transfer between readouts

If

$$\pi: \mathcal{X}_r \to \mathcal{X}_s$$

preserves lower-order witnesses losslessly, it induces higher-order-to-lower-order relation transfer.

Therefore

capacity, lifetime, alphabet, freezing, certificates, complexity, and readout transfer together form the seven derived structures of the relation matrix.

(IV.10)

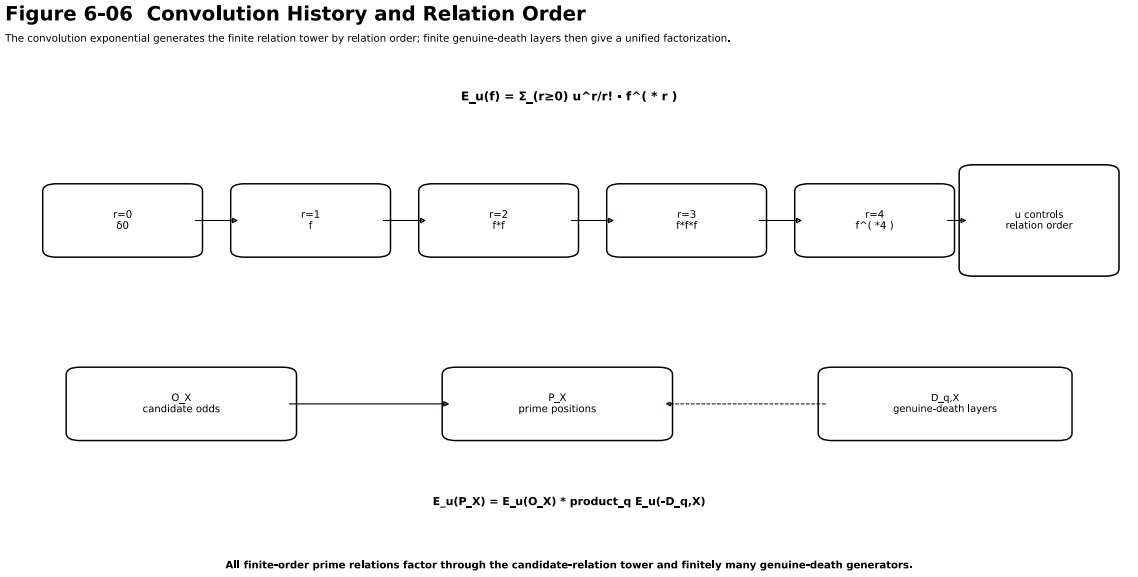


Figure 6-06. Convolution History and Relation Order. The convolution exponential $\mathcal{E}_u(f) = \sum_{r\geq 0} u^r\, f^{*r}/r!$ uses one parameter u to encode unary, binary, ternary, and all finite relation orders, with $\partial_u \mathcal{E}_u(f) = f * \mathcal{E}_u(f)$. In a finite window, uniqueness of genuine death further gives $O_X = P_X \sqcup \bigsqcup_{q\leq\sqrt{X}} D_{q,X}$, thereby factorizing all finite-order prime relations into the candidate-relation tower and finitely many genuine-death generators.

Part V How Higher-Order Relations Transfer Information to Lower-Order Relations

Figure 6-04 Unified Relation Map of Classical Propositions

Columns classify relation geometry and rows classify global quantifiers; named propositions occupy different locations and readouts of one common matrix.

	Fixed sum	Fixed gap	Affine/template	Window/index	Polynomial/exponential	Factor/spectral
Universal fiber	Goldbach			Andrica Firoozbakht		
Infinite recurrence		Twin	Sophie Germain Hardy–Littlewood k-tuple Dirichlet		n^2+1 Bunyakovsky Mersenne	
Window universal / arbitrary length			Green–Tao	Bertrand Legendre Oppermann	Fermat	
Parameter infinite		Polignac		Brocard	Schinzel H	
Identity-block / global spectrum						Grimm Erdős–Woods RH / GRH

A proposition may carry finer geometric and quantifier attributes; the figure marks only its principal readout position.

Figure 6-04. Unified Relation Map of Classical Propositions. Classical prime propositions can be located simultaneously by relation geometry and global quantifier: fixed sums, fixed gaps, affine templates, windows/indices, polynomial or exponential orbits, and factor/spectral geometry cross universal, recurrent, window, arbitrary-length, parameter-family, and spectral readouts.

Chapter 45 Structural Maps Between Relations

Let

$$\pi: \mathcal{X}_r \to \mathcal{X}_s, \qquad s < r.$$

If

$$\pi(\Gamma_F)$$

lands in another relation family, then

$$\boxed{\Gamma_F \to \pi(\Gamma_F)} \tag{45.1}$$

forms a structural map between relations.

Thus the carrying power of the unified matrix means more than

many problems can be written in the same framework.

More strongly,

genuine mathematical information can be transferred between different relation readouts.

(45.2)

Chapter 46 Lossless Higher-Order-to-Lower-Order Transfer

Definition 46.1

Suppose each higher-order relation instance R produces finitely many lower-order relation witnesses

$$\pi_1(R), \dots, \pi_m(R),$$

and each $\pi_j(R)$ can be recovered as concrete lower-order endpoints. We call this a **witness-faithful higher-order-to-lower-order map**.

The mirror folding of a Green–Tao arithmetic progression into a fixed sum is the canonical example.

Theorem 46.2 Delivery of Higher-Order Witnesses

If

$$R \in \mathcal{V}_r$$

and

$$\pi_j(R) \in \mathcal{V}_s$$

for $1 \le j \le m$, then

$$\boxed{R \text{ delivers at least } m \text{ genuine relation witnesses to the lower-order readout.}} \tag{46.1}$$

The proof follows directly from the definition of the map.

Chapter 47 Reverse Readout from the Fixed-Sum Witness Field to Prime Constellations

Part III defined

$$W_K(m) = \{a \in A_K : m - 1 - a \in A_K\}.$$

and proved

$$\boxed{\sum_m \left| \bigcap_{h \in H} W_K(m+h) \right| = K\, T_K(H).} \tag{47.1}$$

Therefore

the same fixed-sum witness field carries the exact finite counts of all finite prime constellations. (47.2)

This is an exact information readout from the lower-order fixed-sum field to fixed-offset patterns.

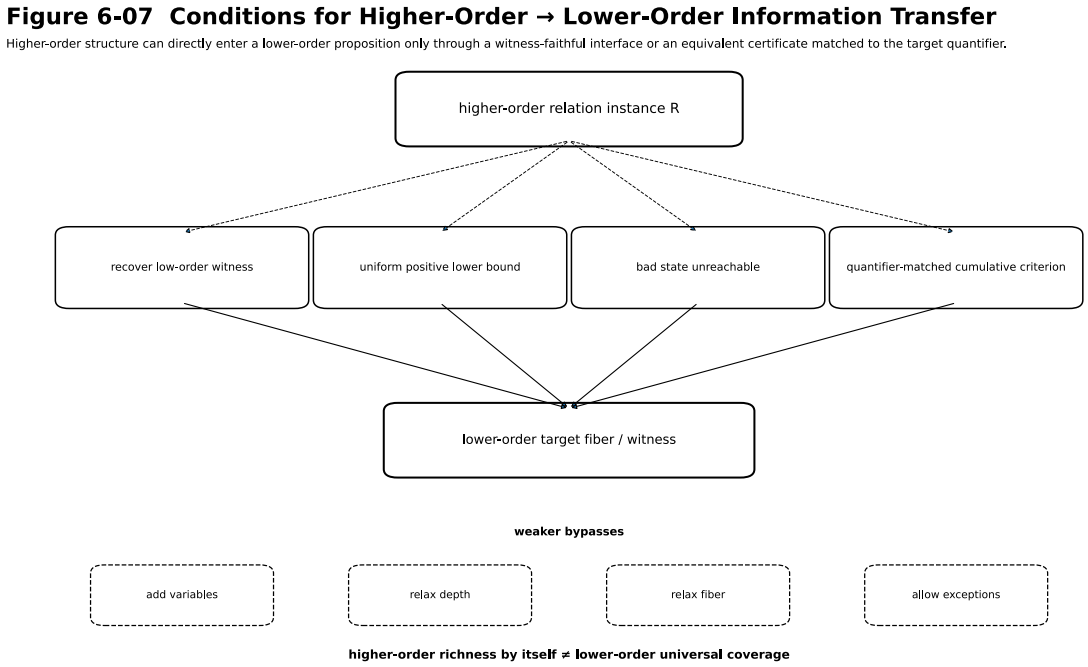


Figure 6-07. Conditions for Higher-Order → Lower-Order Information Transfer. For a higher-order relation to enter a lower-order target directly, it needs at least one witness-faithful interface: recovery of a concrete lower-order witness, a uniform positive lower bound for the target fiber, proof that the lower-order bad state is unreachable, or a cumulative criterion matched to the target quantifier. By contrast, adding variables, relaxing multiplicative depth, relaxing the relation fiber, or allowing an exceptional set may add redundancy but does not automatically recover the universal witnesses required by the original lower-order proposition.

Chapter 48 Higher-Order Existence Does Not Imply Lower-Order Universality

Suppose a class of higher-order relations has infinitely many instances.

Even if every higher-order instance delivers several lower-order witnesses, this shows only that

some lower-order targets receive many witnesses.

It does not automatically imply

every lower-order target has a witness.

Mirror folding from Green–Tao to fixed sums implies that even numbers of unbounded fixed-sum multiplicity continue to occur, but this has a different quantifier from

$$\forall N.$$

Therefore

higher-order abundance cannot replace lower-order universal coverage.

(48.1)

Chapter 49 Why Lower-Order Relations Have the Least Redundancy

Let the relation order be

$$r.$$

As r increases,

- the number of parameter combinations grows;
- the convolution order grows;
- the relation space becomes thicker;
- one local anomaly has less control over the entire relation.

But

$$r = 2$$

is the lowest nontrivial relation order.

Therefore

binary relations are not "naturally easy because their form is simple";
they sit precisely at the point of minimal combinatorial redundancy.

(49.1)

Chapter 50 Four Relaxations Rejected by Lower-Order Propositions

Many methods obtain additional power by using one or more of the following relaxations.

50.1 Increasing the number of variables

Lift a binary relation to a ternary or higher-order relation.

50.2 Relaxing multiplicative depth

Relax

$$P_1$$

to

$$P_2, \quad P_3, \ldots$$

50.3 Relaxing the relation fiber

For example, replace the exact fixed difference

$$h = 1$$

with the requirement that some difference in

$$h \leq H$$

occurs.

50.4 Allowing an exceptional set

Relax

$$\forall N$$

to

almost all N.

Thus

the extra power of higher-order or average methods usually comes from added redundancy;
the original lower-order proposition requires that redundancy eventually be removed.

(50.1)

Chapter 51 Structure May Be Lifted in Order; Witnesses Must Not Be Lost

This principle runs throughout the book:

structure may be lifted in order; witnesses must not be lost.

(51.1)

A legitimate higher-order tool must satisfy at least one of the following:

1. concrete lower-order relation witnesses can be recovered;
2. a uniform positive lower bound for the target lower-order fiber can be proved;
3. lower-order failure states can be shown to be unreachable;
4. a cumulative criterion exactly matching the target quantifier can be given.

Otherwise, a higher-order statistic can serve only as a diagnostic or intermediate quantity.

Part VI The Place of Classical Sieve Methods in the Unified Matrix

Chapter 52 The Sieve of Eratosthenes: Multiplicative Occupancy

The core of the sieve of Eratosthenes is

$$p \mid n.$$

In the language of this book, it corresponds exactly to

$$\boxed{\text{occupancy of integer positions by multiplicative rays.}} \tag{52.1}$$

It answers

$$\boxed{\text{Which positions can no longer be new atoms?}}$$

It lies very close to the lowest-level generative fact of the fixed spectrum.

But pointwise occupancy alone does not preserve

- the first killer;
- multi-endpoint relations;
- relation birth;
- permanent inheritance of relations;
- relation quantifiers.

Thus it is a foundational component of relation theory, not the entire relation structure.

Chapter 53 Brun's Sieve: Multi-Endpoint Rough Relations

Brun's sieve systematically places several related linear forms into the same sieving process. Its natural object is

whether several endpoints simultaneously avoid small prime factors.

Thus it naturally produces

- rough pairs / rough tuples;
- local factors;
- candidate densities;
- upper bounds.

In the unified language, it has already entered the relation layer, but primarily preserves

local multiplicative escape.

Complete relation identity still requires terminal information or additional depth information.

Chapter 54 Selberg's Sieve: Optimizing Local Exclusion Does Not Add Missing Identity Information

Selberg's sieve optimizes the use of local divisibility information through weights.

It answers

how to use the available local exclusion information more effectively.

But if the input does not preserve the exact depth identity

$$(-1)^{\Omega(n)}$$

or

$$\Omega(n) = 1,$$

then no choice of positive weights can recover that missing information from nothing.

Therefore

optimizing how information is used is not the same as adding information that was absent from the input.

(54.1)

Chapter 55 Combinatorial and Linear Sieves: Shallow Multiplicative Depth

When one sieves deeply enough, a survivor's least prime factor is already large.

Hence the total number of prime factors of a survivor is constrained.

This naturally produces shallow multiplicative-depth layers

$$P_1, \quad P_2, \quad P_3, \ldots$$

Thus

combinatorial and linear sieves naturally read "rough survivor + finite multiplicative depth."

(55.1)

Chen-type results are exactly of the form

fixed relation geometry + relaxed target depth.

Chapter 56 The Large Sieve: Multi-Modulus Average Control

The large sieve places many congruence conditions under simultaneous average control.

It is particularly effective for

$$\boxed{\text{multi-modulus distribution} + \text{average exclusion} + \text{global mean-square control.}}$$

But

$$\boxed{\text{average control over many moduli} \neq \text{guaranteed nonemptiness of one specified relation fiber.}} \tag{56.1}$$

It belongs to the statistical layer, not to the terminal-identity layer of a specified relation.

Chapter 57 Bombieri–Vinogradov: Average Distribution Across Moduli

Bombieri–Vinogradov provides very strong average distribution in arithmetic progressions.
It underlies modern sieve methods, bounded-gap results, and many average statements.
Its unified location is

average distribution control of the atomic layer across many congruence sublattices.

Therefore

even very strong average control cannot automatically change the quantifier from “on average” to “for every case.”

(57.1)

Chapter 58 The Hardy–Littlewood Circle Method: Frequency Readout for Fixed Sums

The circle method converts

$$x_1 + \cdots + x_r = N$$

into a Fourier integral.

Its structure is

$$\boxed{\text{major arcs} + \text{minor arcs.}}$$

The major arcs capture the principal local arithmetic structure;

the minor arcs control global oscillation.

Thus the place of the circle method in the unified matrix is

a frequency microscope for fixed-sum relations.

(58.1)

It acts on an already specified fixed-sum relation; it does not redefine primality.

Chapter 59 Why Ternary and Higher-Order Additive Relations Have More Smoothing

Suppose

$$P = M + E.$$

If coarse and fine Fourier supports are strictly separated, then

$$P^{*r} = M^{*r} + E^{*r}.$$

As r increases, the fine residual undergoes more convolutions.

Therefore

higher-order relations naturally gain more convolutional smoothing.

(59.1)

Binary relations are precisely the lowest nontrivial order at which residual smoothing is weakest.

Chapter 60 Chen: Relaxing Depth While Keeping the Relation Fixed

The place of Chen's method in the method spectrum is particularly clear:

the fixed-sum relation remains unchanged

while

$$(1,1)$$

is relaxed to

$$(1, \leq 2)$$

or its symmetric counterpart.

Therefore

the method gains additional sieve margin by relaxing target depth, not by changing relation geometry.

(60.1)

Chapter 61 GPY and Maynard–Tao: Differences Compressed into a Finite Band

These methods compress the gap problem into a finite band:

$$\boxed{\exists H < \infty,}$$

such that some bounded-gap relation recurs infinitely often.

In unified form:

$$\boxed{\text{at least one relation fiber in a finite difference band is infinitely active.}} \tag{61.1}$$

An exact fixed difference instead requires

$$\boxed{\text{the specified } h \text{ itself to be infinitely active.}} \tag{61.2}$$

The distinction lies in the exactness of the relation fiber.

Chapter 62 Green–Tao: Higher-Order Combinatorial Structure

Green–Tao shows that the atomic vertex skeleton of the primes contains arbitrarily long higher-order affine structures.

Its unified position is

ARBITRARILY_LONG.

Through mirror folding, projection, and marginalization, it may also transfer information to lower-order relations.

But exactly what is transferred must be proved by the concrete map.

Chapter 63 Analytic L-Functions: The Global Spectrum

L-function methods read the global spectrum obtained after analytically encoding the entire atomic distribution.

Their strengths include

- globality;
- precise error terms;
- encoding of arithmetic information;
- deep relations between zeros and distribution.

Their unified position is

$$\boxed{\text{fixed spectrum} \to \text{counting measure} \to L\text{-function} \to \text{spectrum.}}$$

They live at a different layer from finite relation terminality.

Chapter 64 Pocklington, Lucas, APR, and ECPP: Finite Certificates

These methods answer

> How can one concrete large integer be finitely proved prime?

They correspond to

finite terminal certification.

The common Lucas certificate for Twin in Part V shows that

> **pointwise certificates can be lifted further to relation-level certificates.**

(64.1)

Thus certificate methods and the relation matrix meet naturally at the layer of finite terminality.

Part VII Primorial Coarse–Fine Decomposition and Analytic Relation Sieves

Chapter 65 The Coarse Space on a Finite Cyclic Group

Take

$$G_L = \mathbb{Z}/L\mathbb{Z},$$

and let

$$W \mid L.$$

Define

$$\boxed{V_W = \{f : G_L \to \mathbb{C} : f(n) \text{ depends only on } n \bmod W\}.} \tag{65.1}$$

Let

$$\Pi_W$$

be the orthogonal projection onto V_W.

For a finite prime field or relation field P, write

$$\boxed{P = M_W + E_W,} \tag{65.2}$$

where

$$M_W = \Pi_W P,$$

$$E_W = (I - \Pi_W)P.$$

Interpret this as

$$\boxed{\text{coarse local periodic skeleton} + \text{fine global remainder.}} \tag{65.3}$$

Chapter 66 Strict Separation of Fourier Supports

If

$$L = WT,$$

define the coarse frequency set

$$\boxed{K_W = \{0, T, 2T, \dots, (W-1)T\}.} \tag{66.1}$$

Then

$$\mathrm{supp}\widehat{M}_W \subseteq K_W,$$

and

$$\hat{E}_W(k) = 0 \qquad (k \in K_W).$$

Hence

$$\boxed{\mathrm{supp}\widehat{M}_W \cap \mathrm{supp}\hat{E}_W = \varnothing.} \tag{66.2}$$

This is stronger than ordinary L^2-orthogonality.

Chapter 67 Exact Decoupling Under Convolution

By multiplication on the Fourier side,

$$\boxed{M_W * E_W = 0.} \tag{67.1}$$

More generally,

$$\boxed{M_W^{*a} * E_W^{*b} = 0, \qquad a, b \geq 1.} \tag{67.2}$$

Therefore

$$\boxed{P^{*r} = M_W^{*r} + E_W^{*r}.} \tag{67.3}$$

Hence

the genuinely non-coarse part that must be controlled is not a collection of complicated mixed terms, but the pure higher convolution of the fine residual.

(67.4)

Figure 6-09 Primorial Coarse-Fine Decomposition

On a finite cyclic group, the coarse periodic skeleton and the fine residual have strictly separated Fourier supports, so mixed convolution terms vanish exactly.

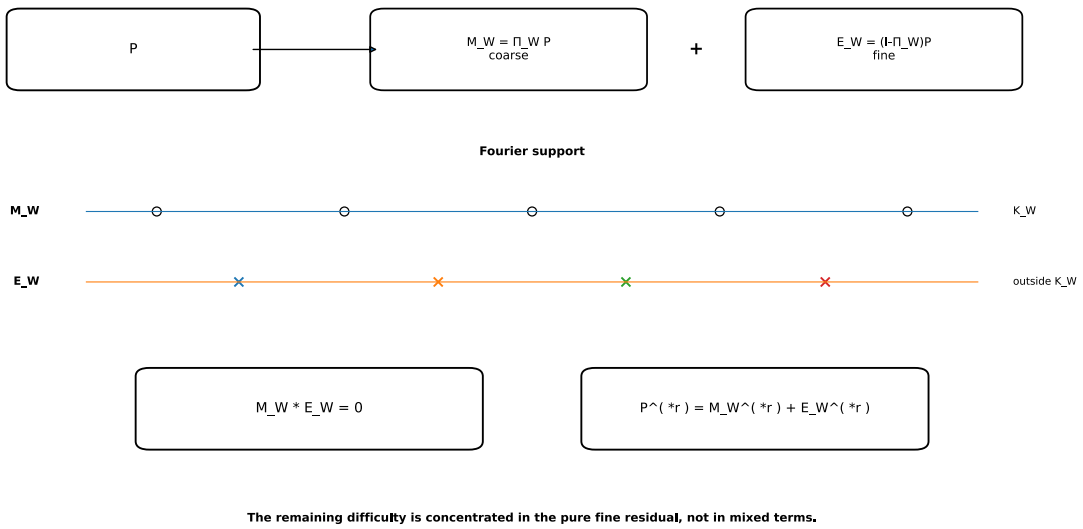


Figure 6-09. Primorial Coarse–Fine Decomposition. On the finite cyclic group G_L, write $P = M_W + E_W$, where $M_W = \Pi_W P$ is the coarse periodic skeleton depending only on residue modulo W, and $E_W = (I - \Pi_W)P$ is the fine residual. Their Fourier supports are strictly disjoint, so $M_W * E_W = 0$; more generally, every mixed convolution containing both coarse and fine factors vanishes, giving $P^{*r} = M_W^{*r} + E_W^{*r}$.

Chapter 68 Primorial Shells

Let

$$W_j = \prod_{i \le j} p_i.$$

Let the corresponding projections be

$$\Pi_j = \Pi_{W_j}.$$

Define

$$\boxed{\Delta_j P = (\Pi_j - \Pi_{j-1})P.} \tag{68.1}$$

The Fourier supports of distinct shells are disjoint, so

$$\boxed{\Delta_i * \Delta_j = 0 \qquad (i \neq j).} \tag{68.2}$$

Each newly added prime modulus contributes a new layer of spectral information.

Chapter 69 Forced Sieve Direction and Transverse Innovation

Write the new shell as

$$\boxed{\Delta_j P = F_j + Y_j,} \tag{69.1}$$

where

- F_j: the component forced by the requirement that one child be forbidden by p_j;
- Y_j: genuine nonuniformity among the admissible children.

Moreover,

$$F_j \perp Y_j.$$

Define

$$\mathcal{F}_j = \| F_j \|_2^2,$$

$$\mathcal{T}_j = \| Y_j \|_2^2.$$

$\mathcal{T}_j$ is the transverse innovation energy.

Chapter 70 Recursion for Coarse Energy

Let

$$\mathcal{C}_j = \| \Pi_j P \|_2^2.$$

Then

$$\boxed{\mathcal{F}_j = \frac{\mathcal{C}_{j-1}}{p_j - 1}.} \tag{70.1}$$

Therefore

$$\boxed{\mathcal{C}_j = \frac{p_j}{p_j - 1}\mathcal{C}_{j-1} + \mathcal{T}_j.} \tag{70.2}$$

Let

$$A_j = \prod_{i \le j} \frac{p_i}{p_i - 1} = \frac{W_j}{\varphi(W_j)}.$$

Define

$$\boxed{\mathcal{H}_j = \frac{\mathcal{C}_j}{A_j}.} \tag{70.3}$$

Then

$$\boxed{\mathcal{H}_j - \mathcal{H}_{j-1} = \frac{\mathcal{T}_j}{A_j} \ge 0.} \tag{70.4}$$

Hence

$$\boxed{\mathcal{H}_J - \mathcal{H}_k = \sum_{j=k+1}^{J} \frac{\mathcal{T}_j}{A_j}.} \tag{70.5}$$

This is an exact telescope for transverse energy.

Chapter 71 Transport of Local Factors

Let

$$R_{r,j}(N)$$

denote the mass of an order-r relation at the j-th primorial scale.

Purely local refinement contributes the multiplier

$$\sigma_{r,p_j}(N).$$

For a binary fixed-sum relation,

$$\boxed{\sigma_{2,p}(N) = \begin{cases} \dfrac{p}{p-1}, & p \mid N, \\ 1 - \dfrac{1}{(p-1)^2}, & p \nmid N. \end{cases}} \tag{71.1}$$

Let the discrepancy between the actual refinement and the completely uniform allowed-child model be denoted by

$$\varepsilon_{r,j}(N).$$

Then

$$\boxed{R_{r,j}(N) = \sigma_{r,p_j}(N) R_{r,j-1}(N) + \varepsilon_{r,j}(N).} \tag{71.2}$$

Chapter 72 Finite Duhamel Formula

Iterating from scale k to scale J gives

$$\boxed{R_{r,J}(N) = R_{r,k}(N) \prod_{\ell=k+1}^{J} \sigma_{r,p_\ell}(N) + \sum_{m=k+1}^{J} \varepsilon_{r,m}(N) \prod_{\ell=m+1}^{J} \sigma_{r,p_\ell}(N).} \tag{72.1}$$

This is a completely finite, zero-error identity.

It separates all information into

$$\boxed{\text{explicit local sieve geometry} + \text{transverse forcing.}} \tag{72.2}$$

Part VIII Parity, Child Modes, and the Boundary of Binary Relations

Chapter 73 Parity of Multiplicative Depth

Define

$$\boxed{\lambda(n) = (-1)^{\Omega(n)}.} \tag{73.1}$$

A positive sieve primarily preserves the information

whether a point has escaped the small prime factors.

By contrast, λ preserves

the parity of multiplicative depth.

Therefore

$$\boxed{\text{parity barrier} = \text{loss, under purely positive sieve information, of the nontrivial multiplicative-de}} \tag{73.2}$$

Chapter 74 Rough Survivors and Signed Survivors

Let

$$R_z$$

denote the z-rough survivor field, and let

$$L_z$$

denote the survivor field signed by λ.

When the surviving multiplicative depths have been reduced to

$$1, \quad 2,$$

the two channels can be recovered through

$$\boxed{\begin{pmatrix} P_z \\ Q_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} R_z \\ L_z \end{pmatrix}.} \tag{74.1}$$

This recovers the prime and P_2 channels.

Thus

adding one signed depth channel recovers part of the identity information lost by a purely positive sieve.

(74.2)

Chapter 75 Parity-Resolved Decomposition of a Fixed Sum

Define

$$\mathsf{G}_z(N) = (P_z * P_z)(N).$$

Since

$$P_z = \frac{1}{2}(R_z - L_z),$$

we obtain

$$\boxed{4\mathsf{G}_z(N) = (R_z * R_z)(N) - 2(R_z * L_z)(N) + (L_z * L_z)(N).} \tag{75.1}$$

Define

$$A_z(N) = (R_z * R_z)(N),$$

and

$$\boxed{\mathfrak{P}_G(N;z) = 2(R_z * L_z)(N) - (L_z * L_z)(N).} \tag{75.2}$$

Then

$$\boxed{4\mathsf{G}_z(N) = A_z(N) - \mathfrak{P}_G(N;z).} \tag{75.3}$$

If

$$A_z(N) > 0,$$

define

$$\boxed{\Theta_G(N;z) = \frac{\mathfrak{P}_G(N;z)}{A_z(N)}.} \tag{75.4}$$

Then

$$\boxed{\mathsf{G}_z(N) > 0 \Leftrightarrow \Theta_G(N;z) < 1.} \tag{75.5}$$

Extinction of the fixed-sum relation is exactly the condition

$$\boxed{\Theta_G = 1.} \tag{75.6}$$

Chapter 76 The Absorbing Nature of Parity Saturation

If a target has reached

$$\Theta_G = 1,$$

then the corresponding prime-prime relation mass is zero.

Under purely positive sieve refinement, this state is not an unstable state that is automatically restored.

The correct structural statement is

$$\boxed{\text{parity saturation is an absorbing state.}} \tag{76.1}$$

Thus the genuine question is

$$\boxed{\text{when can the first extinction occur?}} \tag{76.2}$$

Chapter 77 First Extinction and Vertex Covers of the Relation Graph

Let the odd-depth survivor set before one filtering step be

$$O.$$

A new prime p genuinely deletes

$$K_p^\sharp \subset O.$$

Represent the fixed-sum relation as the graph

$$x + y = N.$$

The new layer extinguishes all old relations for the target N if and only if

$$\boxed{K_p^\sharp \text{ is a vertex cover of the old relation graph.}} \tag{77.1}$$

In the typical case

$$p \nmid N,$$

the two endpoints of a relation edge cannot both be genuinely deleted by p.

Hence first extinction requires

exactly one endpoint of every old relation edge to enter the killer set. (77.2)

This converts abstract parity saturation into a concrete collapse of relation-residue structure.

Chapter 78 Ordinary Death and True Death

In an ordinary divisibility sieve,

$$p \mid n$$

is commonly treated as a hit at the current layer.

But

$$n = p$$

is itself a newly born atom, not a death.

Therefore

$$\boxed{\text{ordinary death} \neq \text{true death.}} \tag{78.1}$$

A relation sieve must use

$$P^{-}(n) = p$$

to record the genuine first killer.

This is the common source of the genuine-death structures used in Parts IV and V.

Chapter 79 Child-Mode Anisotropy

Under the refinement

$$W_{j-1} \to W_j = W_{j-1}p_j,$$

write the transverse field as

$$Y(A,B),$$

where A is the parent coordinate and B is the child coordinate.

Define

$$\boxed{S_B(A) = (Y * Y)(A,B).} \tag{79.1}$$

Its child Fourier transform is

$$\boxed{Q_\xi(A) = \sum_{B \bmod p} S_B(A) e_p(-\xi B).} \tag{79.2}$$

Because the transverse field has no child zero mode,

$$Q_0(A) = 0.$$

Hence

$$\boxed{\sum_{B \bmod p} S_B(A) = 0.} \tag{79.3}$$

Define

$$\overline{Q}(A) = \frac{1}{p-1}\sum_{\xi \neq 0} Q_\xi(A),$$

and

$$\boxed{\mathfrak{A}_p(A)^2 = \frac{1}{p-1}\sum_{\xi \neq 0} |Q_\xi(A) - \overline{Q}(A)|^2.} \tag{79.4}$$

Parseval's identity gives

$$\boxed{\mathfrak{A}_p(A)^2 = \frac{p}{p-1}\sum_{B \neq 0} \left| S_B(A) + \frac{S_0(A)}{p-1} \right|^2.} \tag{79.5}$$

Thus

child-mode anisotropy is precisely the variance in relation allocation among the distinct nonzero child targets.

(79.6)

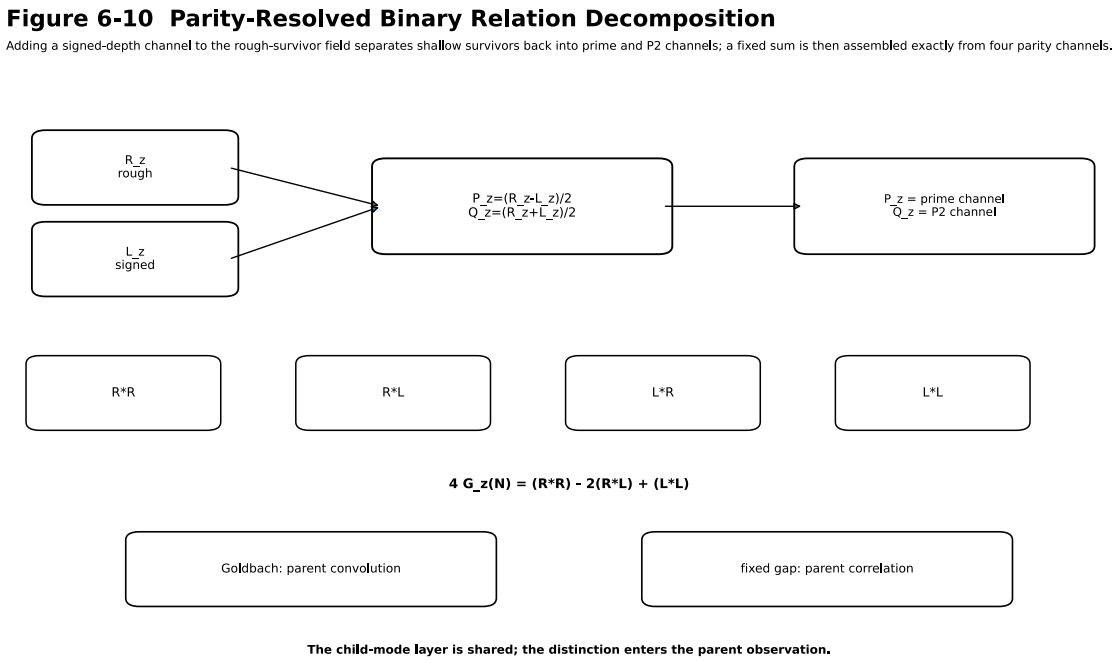


Figure 6-10. Parity-Resolved Binary Relation Decomposition. When rough survivors have multiplicative depth only 1 or 2, the rough field R_z together with the signed field L_z carrying $\lambda = (-1)^{\Omega}$ recovers the prime channel $P_z = (R_z - L_z)/2$ and the P_2 channel $Q_z = (R_z + L_z)/2$. For a fixed sum, $4\mathrm{G}_z = (R_z * R_z) - 2(R_z * L_z) + (L_z * L_z)$. Goldbach and fixed differences share the child-mode layer; the distinction enters the parent convolution versus correlation readout.

Chapter 80 Analytic Duality Between Goldbach and Fixed Differences

Goldbach reads the parent convolution

$$\tilde{Y}_{\xi} * \tilde{Y}_{\xi}.$$

A fixed difference reads the parent correlation

$$\tilde{Y}_{\xi} \star \tilde{Y}_{\xi}.$$

The child-mode layer is the same.

Therefore

Goldbach and fixed even differences share the same child-mode anisotropy; the distinction enters only through the parent observation: one is convolution, the other correlation.

(80.1)

This is exactly consistent with the sum/difference dual slices of the binary relation plane developed in Part III, now read through Fourier-child language.

Chapter 81 The Distance Between Average Control and All-Target Control

For a binary residual, a fourth moment can control the number of exceptional targets.

A typical structure is

$$\sum_{N} |E_X(N)|^2 \le \int_{\mathfrak{m}} |S_X(\alpha)|^4 \, d\alpha.$$

Such tools may imply

$$|\mathcal{E}(X)| = o(X)$$

or even stronger exceptional-set bounds.

But

the proportion of exceptional targets tending to zero ⇒/ the exceptional set being empty. (81.1)

Therefore

there is a genuine quantifier gap between average structure and a universal relation.

(81.2)

Chapter 82 The Information Boundary of Pure Positive Energy

Coarse energy, transverse energy, L^q -norms, fourth moments, and related absolute-value quantities primarily describe

how much oscillation or energy is present.

Multiplicative-depth parity, by contrast, describes

which multiplicative-depth channel that mass belongs to.

Thus

energy describes `how much''; a parity character describes` **which class.''**

(82.1)

Purely positive energy alone cannot recover a lost depth sign.

Chapter 83 Boundary of a Single-Scale Reflection Gap

A single prime scale can exhibit a highly coherent reflection structure.

Therefore one cannot require

a uniform positive reflection gap at every layer.

The correct cross-scale question is

can high coherence persist across many prime scales?

(83.1)

This is precisely where relation histories, cross-scale accumulation, and child-mode structure become relevant.

Chapter 84 Moving-Target Determinant Structure

Bilinear methods for a fixed shift often exploit a small-determinant structure.

For a fixed-sum moving target one encounters

$$\boxed{bmn - ars = N(b - a).} \tag{84.1}$$

Here

$$N \asymp x$$

itself moves with the target.

Thus the small-determinant structure of a fixed-shift problem cannot simply be transferred unchanged to a moving-target fixed-sum relation.

This shows that

the suitability of a tool is determined by relation geometry, not merely by superficial similarity of formulas.

(84.2)

Chapter 85 RWF-II as an Analytic Interface

Near

$$z = N^{1/4-\varepsilon},$$

a rough odd-depth survivor can be reduced to the structure

$$P_1 \cup P_3.$$

A reflection-type well-factorable Type-II target has the form

$$\boxed{\sum_{d,m} \omega_d^- \lambda(d)\lambda(m)\Lambda(N - dm) = o(N/\log z).} \tag{85.1}$$

where

$$D = N^{1/2-o(1)}.$$

The significance of such an estimate is that it

compresses the deep residual of a fixed-sum moving target into an extremely narrow multiplicative-depth structure.

It is an analytic interface; it does not alter the underlying relation matrix.

Part IX Master Table of Method Capability Boundaries

Chapter 86 Methods and Information Levels

Method	Information primarily preserved	Most natural output
Sieve of Eratosthenes	single-point multiplicative occupancy	primary prime/composite separation
Brun sieve	multi-end rough escape	upper bounds and candidates for pairs/tuples
Selberg sieve	optimized local exclusion	weighted sieve bounds
combinatorial sieve / linear sieve	roughness + shallow depth	P_r results
large sieve	average distribution over many moduli	mean-square and average control
Bombieri–Vinogradov theorem	average distribution in arithmetic progressions	strong average-level control
circle method	fixed-sum Fourier structure	main terms and errors for higher-order additive relations
Chen's method	fixed sum + relaxed depth	prime + P_2
GPY / Maynard–Tao	finite gap band	bounded gaps
Green–Tao	higher-order affine structure	arbitrarily long arithmetic progressions
L-functions	global spectrum	density, error terms, and zero relations
Pocklington / Lucas / APR / ECPP	finite identity of a single point	verifiable primality certificates
relation-level Lucas	common group order across multiple endpoints	finite relation certificates

The point of this table is not to rank methods by strength.

It answers the question

exactly which kind of information does a method preserve?

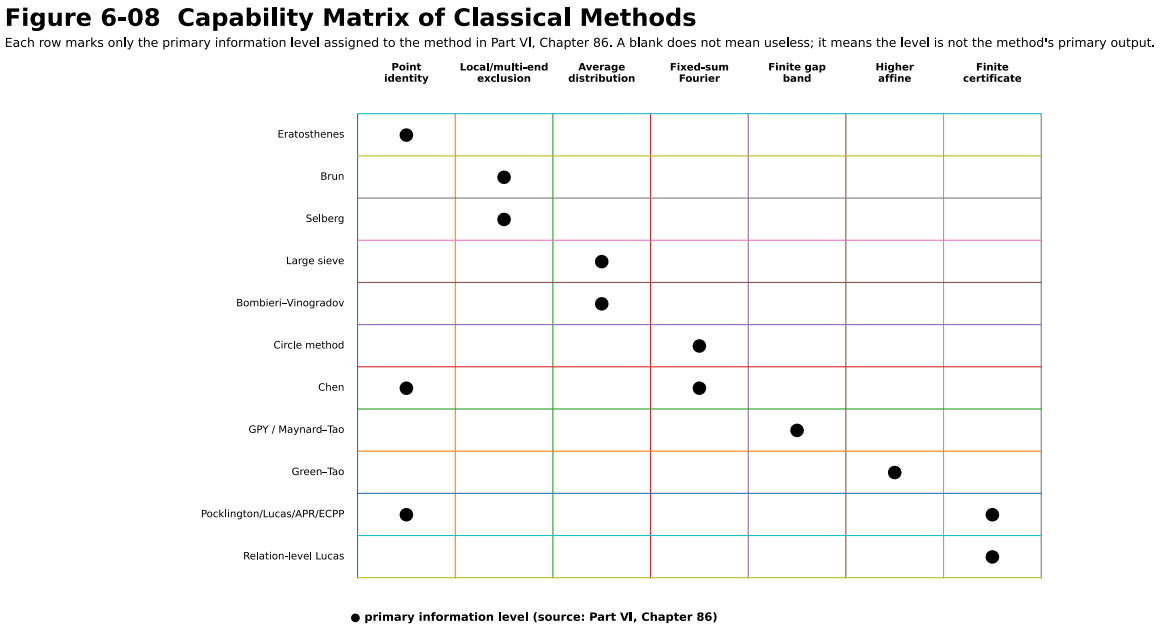


Figure 6-08. Capability Matrix of Classical Methods. The matrix marks only the primary information level preserved or controlled by each classical method in the capability table of Part VI. Sieves, average-distribution tools, the circle method, finite-gap methods, higher-order affine methods, and finite certificates have distinct primary outputs; unifying the relation object does not force every problem to use one tool.

Chapter 87 When Can a Higher-Order Method Enter a Lower-Order Proposition?

For a higher-order method to enter a lower-order target directly, there must be an explicit interface

$$\boxed{\text{higher-order object} \to \text{lower-order witness.}}$$

Possible interfaces include:

1. an explicit projection;
2. mirror folding;
3. an exact counting identity;
4. a uniform positive lower bound;
5. exclusion of bad states;
6. a cumulative lower bound matched to the target quantifier.

If no such interface exists, the higher-order structure may still be important, but it belongs to another level.

Chapter 88 Almost-All and All-Target

“Almost all” means that the exceptional set is relatively sparse.

“Every target” means that the exceptional set is empty.

Therefore

almost-all is a density statement; all-target is an emptiness statement.

(88.1)

Passing from the former to the latter requires additional structure capable of excluding the final exceptional points.

This cannot be replaced by the statement that the exceptional proportion is already extremely small.

Chapter 89 Bounded Band and Fixed Fiber

Suppose one proves that

$$\exists h \leq H$$

for which a gap relation recurs infinitely often. This yields a finite-band result.
If the target is instead that

$$h = h_0$$

itself recurs infinitely often, then the relation fiber must be fixed at h_0.
Hence

existence in a finite band cannot replace persistence in a specified fiber.

(89.1)

Chapter 90 Almost-Primes and Prime Depth

If a method yields only

$$\Omega(n) \leq 2,$$

that does not automatically replace

$$\Omega(n) = 1.$$

Thus

shallow multiplicative depth is not synonymous with prime depth.

(90.1)

Nevertheless, shallow-depth results remain extremely valuable because they locate precisely which depth layers of the matrix have already been reached by the relation.

Figure 6-11 Average Control vs All-Target Control

Three quantifier gaps: almost-all is not all-target; a bounded band is not a fixed fiber; an almost-prime is not prime depth.

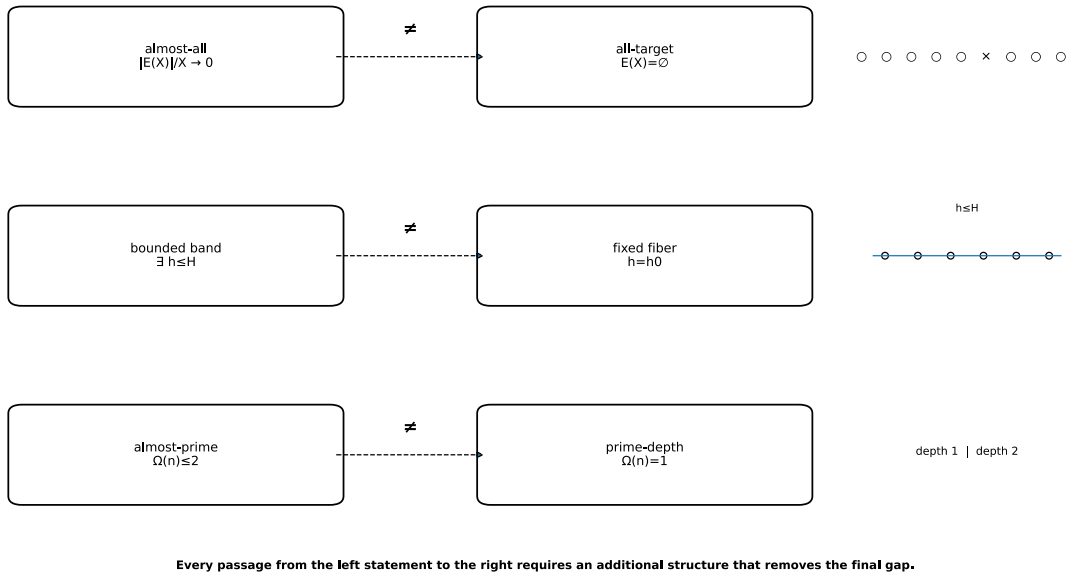


Figure 6-11. Average Control vs All-Target Control. Part VI emphasizes three quantifier gaps that must not be conflated: an exceptional-set density tending to zero does not imply the exceptional set is empty, so almost-all is not all-target; infinite recurrence for some $h \leq H$ does not imply infinite recurrence in the specified fiber $h = h_0$; and $\Omega(n) \leq 2$ is not the same as prime depth $\Omega(n) = 1$. Passing from each statement on the left to the target on the right requires additional structure that removes the final exceptional, bandwidth, or depth gap.

Chapter 91 Higher-Order Smoothing and Lower-Order Rigidity

Higher-order additive relations often benefit from repeated convolution smoothing in

$$P^{*r}.$$

A binary relation has only

$$P * P.$$

Therefore

higher-order relations more readily acquire average smoothing; lower-order relations expose extreme phase behavior more directly.

(91.1)

This is not a ranking of methods but a structural difference created by the order of the relation itself.

Part X Generative Complexity

Chapter 92 How Should the Difficulty of Relations Be Compared?

Define the generative complexity of a finite relation object R by

$$\boxed{\mathsf{C}_{\mathrm{gen}}(R) = \inf\{\text{finite historical cost sufficient to generate and terminally certify } R\}.} \tag{92.1}$$

Decompose it as

$$\boxed{\mathsf{C}_{\mathrm{gen}} = (\mathsf{C}_{\mathrm{layer}}, \mathsf{C}_{\mathrm{horizon}}, \mathsf{C}_{\mathrm{relation}}, \mathsf{C}_{\mathrm{certificate}}).} \tag{92.2}$$

Chapter 93 Atomic-Layer Complexity

$$\mathsf{C}_{\mathrm{layer}}$$

records how many multiplicative control layers must be advanced.

For example, for a finite relation whose largest endpoint is H, the most elementary certification needs only

$$q \le \sqrt{H}.$$

Stronger structure can sometimes produce earlier termination.

Chapter 94 Horizon Complexity

$$C_{\text{horizon}}$$

records the numerical scale or window width required by the target.

One must distinguish

$$\boxed{\text{coordinate magnitude} \quad \text{from} \quad \text{window width.}} \tag{94.1}$$

A window of width 10^6 located near 10^{18} is a completely different computational object from a dense scan from 0 to 10^{18}.

Chapter 95 Relation Complexity

$$C_{\text{relation}}$$

records:

- the order of the relation;
- the number of endpoints;
- the complexity of the position maps;
- the relation alphabet;
- matching or window structure;
- possible death types;
- the quantifier type.

Chapter 96 Certificate Complexity

$$C_{certificate}$$

records how the final relation is compressed into an independently verifiable certificate.

Examples include:

- trial division for small integers;
- Lucas–Lehmer;
- Pocklington;
- ECPP;
- common Lucas relation certificates;
- matching certificates.

Thus the “difficulty” of different problems can be decomposed into several distinct costs rather than compared only by the names of the problems.

Figure 6-12 Four Layers of Generative Complexity

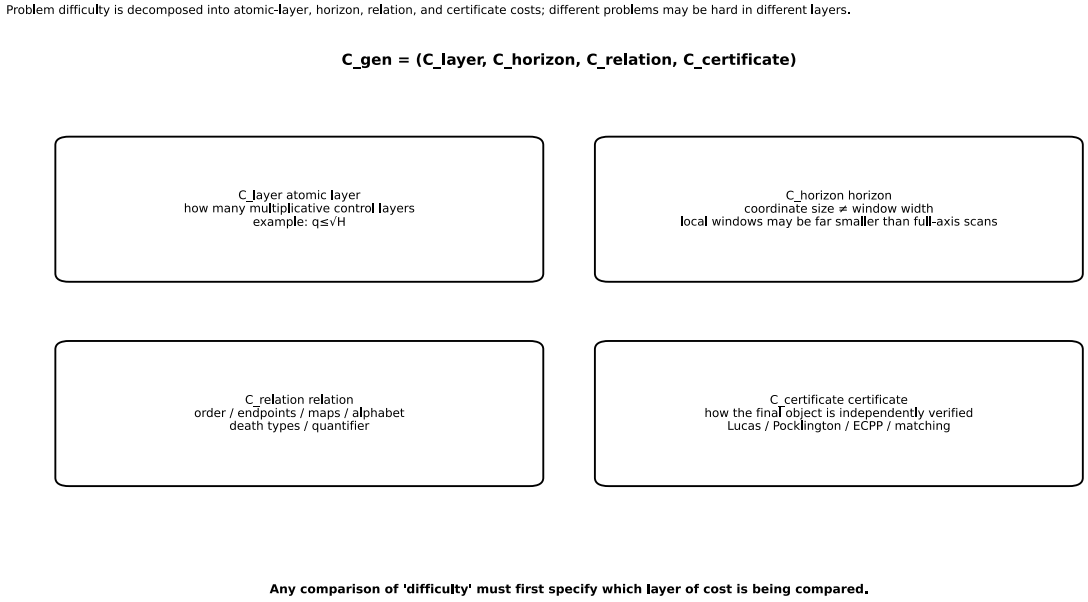


Figure 6-12. Four Layers of Generative Complexity. Generative complexity is written as $C_{gen} = (C_{layer}, C_{horizon}, C_{relation}, C_{certificate})$. The four components record how many multiplicative control layers are required, the numerical horizon or window width, the relation order together with position/alphabet/quantifier structure, and the final independently verifiable certificate. Hence the ‘difficulty’ of different problems cannot be compared by problem name alone.

Part XI Unified Master Tables of Classical Relations

Chapter 97 Master Table of Relation Positions

Relation	Orbit / window	Target identity	Quantifier type
Goldbach	$p, N-p$	P_1, P_1	universal fiber
Lemoine/Lévy	$p, (N-p)/2$	P_1, P_1	universal fiber
weak Goldbach	$p_1+p_2+p_3=N$	P_1^3	universal fiber
Chen	$p+q=N$	$P_1, P_{\leq 2}$	universal type
Twin	$p, p+2$	P_1, P_1	infinite recurrence
Polignac	$p, p+2h$	P_1, P_1 + adjacency condition	infinite recurrence
Sophie Germain	$p, 2p+1$	P_1, P_1	infinite recurrence
prime tuple	$n+H$	P_1^k	infinite recurrence
Green–Tao	AP manifold	P_1^k	arbitrary length
Dirichlet	$a+qn$	P_1	infinite recurrence
Bertrand	$(x, 2x)$	window contains P_1	window universality
Legendre	$(n^2, (n+1)^2)$	window contains P_1	window universality
Oppermann	two square half-windows	each contains P_1	window universality
Brocard	prime-square window	P_1 count	universal predicate
Andrica	(p_n, p_{n+1})	gap metric	universal predicate
Firoozbakht	(p_n, p_{n+1})	normalized growth	universal predicate
n^2+1	quadratic orbit	P_1	infinite recurrence
Bunyakovsky	polynomial orbit	P_1	infinite recurrence
Schinzel H	polynomial family	P_1^k	infinite recurrence
Mersenne	2^p-1	P_1	parameter infinity / infinite recurrence
Fermat	$2^{2^n}+1$	P_1	sparse orbit
Waring–Goldbach	power-sum manifold	P_1^r	fixed-sum type
Grimm	block of consecutive composites	factor-set matching	universal predicate
Erdős–Woods	factor-identity block	block separation	block separation
PNT	natural axis	prime density	density
PNT in AP	congruence sublattice	prime density	density
RH/GRH	L-function spectrum	global oscillation	spectral

Chapter 98 Master Table of Landau's Four Problems

Goldbach	fixed sum
Twin	fixed difference
Legendre	square window
n^2+1	quadratic orbit

(98.1)

These four problems show that

the essence of Unified Prime Relations Theory is not that all problems look alike, but that distinct geometries can be expressed losslessly within the same fixed-identity matrix.

Chapter 99 Standard Procedure for Entering a New Relation into the Matrix

For any new prime relation, answer the following questions in order:

1. Which fields are read from the fixed spectrum?
2. What is the order of the relation?
3. What are the position maps?
4. What is the relation orbit?
5. Which multiplicative depths are allowed?
6. Is there any complete local obstruction?
7. What is the natural finite observation region?
8. Where is the terminal horizon?
9. How is genuine death recorded?
10. How is relation supply defined?
11. How is absorption capacity defined?
12. Can higher-order structure recover a lower-order witness?
13. What is the outer quantifier?
14. What is the corresponding batch closer?
15. How is the final certificate independently verified?

Thus

$$\text{fixed spectrum} \to \text{relation orbit} \to \text{target depth} \to \text{local obstruction} \to \text{finite horizon} \to \text{relation s} \tag{99.1}$$

Figure 6-13 Standard Procedure for Entering a New Relation into the Matrix

The 15 questions of Chapter 99 compress into eight standard workflow nodes that can be reused as a research template for a new relation.

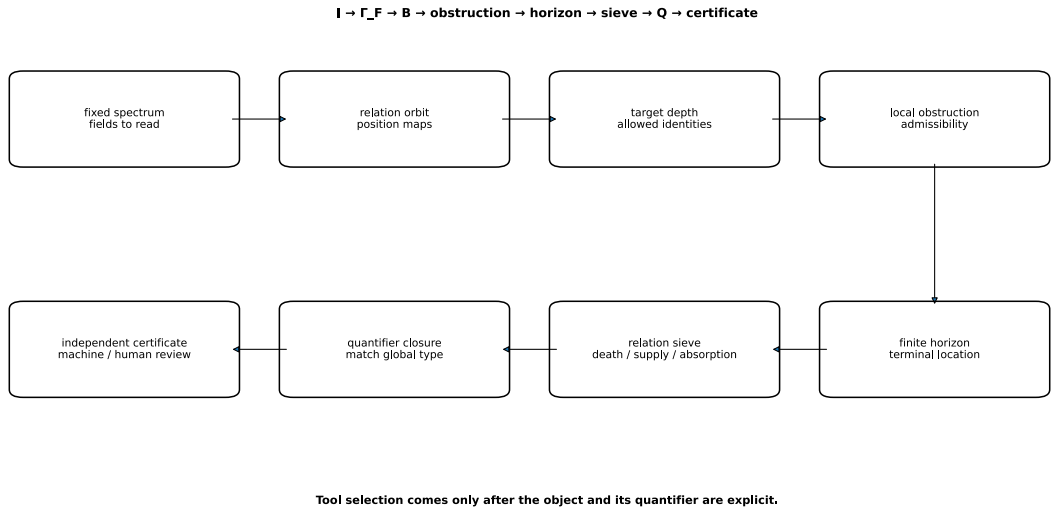


Figure 6-13. Standard Procedure for Entering a New Relation into the Matrix. The fifteen checks of Chapter 99 compress into a standard workflow: fixed-spectrum readout → relation orbit

and position maps → target multiplicative depth → local obstruction/admissibility → finite horizon and terminality → genuine death, supply, and absorption in the relation sieve → outer-quantifier closure → independent certificate. Concrete tools should be selected only after the relation object and quantifier type are explicit.

Chapter 100 Unify the Object Layer, Not the Concrete Tools

The unified matrix fixes

$$\boxed{\text{fixed identity} + \text{relation generation} + \text{persistent history} + \text{finite certification} + \text{relation readout}} \quad (100.1)$$

It does not require every problem to use the same upper-level tool.

Different relations may use:

- relation sieves;
- combinatorial sieves;
- the circle method;
- the large sieve;
- L-functions;
- combinatorial structures;
- finite certificates;
- specialized matching procedures;
- specialized large-integer algorithms.

But all these methods act on already specified relation objects.

Hence

the underlying objects and relational language are unified; upper-level methods remain free.

(100.2)

This is not a compromise.

It is the proper stratification of a mature unified theory.

Master Table of Formal Proofs of Classical Propositions

Classical proposition	Relational normal form	Quantifier / conclusion interface
Goldbach	$G(m) > 0$	universal fiber
Lemoine / Lévy	$L(N) > 0$	universal fiber
weak Goldbach	nonempty ternary fixed sum	universal fiber
Chen	prime + depth ≤ 2	eventual universal fiber
Twin	$C_T(X)$ unbounded	infinite recurrence
Polignac	$P_h(p) = 1$ recurs infinitely often	fixed-gap recurrence
Sophie Germain	$SG(p) = 1$ recurs infinitely often	infinite recurrence
prime k-tuple	$C_H(X)$ unbounded	infinite recurrence
Green–Tao	$\forall k\ \exists a, d$	arbitrarily long
Dirichlet	$C_{a,q}(X)$ unbounded	infinite recurrence
PNT / PNT in AP	prime-counting asymptotic	statistical
Bertrand	$W_B(x) \geq 1$	window universal
Legendre	$W_L(n) \geq 1$	window universal
Oppermann	$W_-(n), W_+(n) \geq 1$	double-window universal
Brocard	$B(n) \geq 4$	universal predicate
Andrica	$A_n < 1$	universal predicate
Firoozbakht	normalized logarithm strictly decreasing	universal predicate
$n^2 + 1$	$C_2(X)$ unbounded	infinite recurrence

Classical proposition	Relational normal form	Quantifier / conclusion interface
Bunyakovsky	$C_f(X)$ unbounded	infinite recurrence
Schinzel H	$C_f(X)$ unbounded	infinite recurrence
Mersenne	$P_M(X)$ unbounded	parameter infinite
infinitely many Fermat primes	$C_F(X)$ unbounded	parameter infinite
Waring–Goldbach	$\mathcal{W}_{k,r}(N) \neq \varnothing$	universal fiber on target domain
Grimm	bipartite matching of factor sets	universal predicate
Erdős–Woods	B_K injective	block separation
RH / GRH	zeros satisfy $\Re s = 1/2$	global spectral

Therefore

every classical proposition can enter the same object–relation–quantifier language losslessly while retaining its irreducible geometry, quantifier, and certificate type.

(T.F)

Figure 6-14 Visual Index of the Classical-Proposition Master Table

Columns are the six relation geometries and rows are seven principal quantifier/readout classes; this is a navigation index, not a replacement for the full definitions in the text.

	Fixed slices	Higher-order affine	Nonlinear orbits	Sparse exponential	Interval / metric	Statistical / spectral
Universal fiber	Goldbach Lemoine/Lévy weak Goldbach Chen		Waring–Goldbach			
Infinite recurrence	Twin Polignac	Sophie Germain prime tuple Dirichlet	n^2+1 Bunyakovsky	Mersenne		
Window universal					Bertrand Legendre Oppermann	
Universal predicate					Brocard Andrica Firoozbakht	Grimm
Arbitrary length		Green–Tao				
Parameter / block			Schinzel H	Fermat		Erdős–Woods
Density / spectral						PNT PNT in AP RH / GRH

A proposition may have additional attributes; the figure marks its principal geometry × quantifier position.

Figure 6-14. Visual Index of the Classical-Proposition Master Table. This figure reorganizes the named relations of Chapter 97 into a navigation index using six principal relation geometries and seven principal quantifier/readout positions. Goldbach, Twin, Legendre, Green–Tao, Bunyakovsky, Mersenne, Grimm, Erdős–Woods, PNT, RH/GRH, and the other listed propositions can therefore be located in one map. A proposition may still have additional geometric or quantifier attributes; the figure marks only its principal position.

Conclusion Classical Problems Are No Longer Isolated, and Classical Methods Are No Longer in Conflict

One reason prime-number theory appears so fragmented is that different problems have long been written in different languages:

- some as additive problems;
- some as difference problems;
- some as interval problems;
- some as polynomial problems;
- some as exponential problems;
- some in terms of factor sets;
- some as Fourier integrals;
- some through L-functions;
- some as primality certificates.

If one looks only at these surface languages, they can easily appear to belong to different worlds.

Unified Prime Relations Theory returns them to the same level.

First there is a single fixed arithmetic identity spectrum

$$\boldsymbol{I}.$$

Upon it sits the atomic layer

$$\mathcal{A}.$$

The atoms form a persistent relation field

$$\Gamma.$$

Each concrete problem then selects

$$\Gamma_F, \qquad B, \qquad \mathcal{R}, \qquad Q.$$

Thus

> **a new prime relation does not redefine the primes; on the same already generated relation matrix, it specifies a relation orbit, multiplicative depth, observation region, and global requirement.**

For the first time, this allows classical problems to be compared at the same object level.

Goldbach and Twin share the binary atomic relation plane:

> one reads the sum coordinate, the other the difference coordinate.

Chen's theorem and Goldbach share the same relation geometry:

what changes is the target depth.

Green–Tao and prime tuples share a higher-dimensional affine structure:

> one uses a fixed pattern; the other permits patterns of arbitrary length.

Legendre, Oppermann, and Brocard share a window language:

they differ in the window generator and the window predicate.

Bunyakovsky, Schinzel H, Mersenne, and Fermat share an orbit language:

they differ in linear, polynomial, exponential, and double-exponential orbits.

Grimm and Erdős–Woods show that

the unified matrix does not merely read prime/composite identity; complete factor sets themselves are part of fixed arithmetic identity.

Classical propositions likewise no longer remain at the level of "they can be viewed this way."

Goldbach has been proved equivalent to

$$A \boxplus A = \mathbb{Z}_{\geq 3}$$

and to coverage of the natural terminal segments, and its universal closure was completed in Part IV;

Twin has been proved equivalent to

$C_T(X)$ being unbounded

and to persistent hits in arbitrarily distant square horizons, and its infinite-recurrence closure was completed in Part V;

window propositions are equivalent to positivity of their window counts;

polynomial and exponential propositions are equivalent to unbounded cumulative counts on the corresponding orbits;

Grimm is equivalent to bipartite matching of factor sets;

Erdős–Woods is equivalent to injectivity of the fixed-length identity-block map.

Therefore

formal proof is the mathematical proof that the original proposition has lost nothing when translated into the unified language.

Classical methods also acquire their precise positions.

The sieve of Eratosthenes sees

multiplicative occupancy.

Brun, Selberg, and combinatorial sieves see

local escape and shallow multiplicative depth.

The large sieve and Bombieri–Vinogradov see

average congruence distribution.

The circle method sees

fixed-sum frequencies.

GPY and Maynard–Tao see

finite gap bands.

Green–Tao sees

higher-order affine structure.

L-functions see

the global spectrum.

Pocklington, Lucas, APR, and ECPP see

finite terminal certificates.

These methods do not cancel one another.

They lead to different types of conclusions because they preserve different types of information.

Thus the most important methodological sentence of this part is

the capability boundary of a method is determined first by the information it preserves.

And the most important discipline governing higher and lower order is

higher-order structure may add redundancy; a lower-order proposition must still recover a lower-order witness.

Together these two statements explain why

almost-all

cannot automatically become

all-target;

bounded band

cannot automatically become

fixed fiber;

P_2

cannot automatically become

P_1;

and

higher-order existence

cannot automatically become

lower-order universality.

But whenever a lossless map exists, higher-order structure can genuinely transmit information to a lower-order problem.

The mirror folding in Green–Tao and the multicolumn persistence identity of the fixed-sum witness field already provide two clear examples.

Thus unification does not mean forcing every problem into one form.

True unification is

one matrix in which distinct relations retain their own geometries, distinct methods retain their freedom, and all objects, witnesses, and quantifiers nevertheless connect exactly in a common language.

At this point the mathematical backbone of the book has formed six layers:

Part I: fixed identities and the atomic relation matrix
⇓
Part II: finite moving horizons and relation reconstruction
⇓
Part III: relation slices, derived geometry, and quantifier structures
⇓
Part IV: CRT survival trees, genuine death, and relation sieves
⇓
Part V: finite certificates, supply–absorption, and infinite quantifiers
⇓
Part VI: formal proofs of classical propositions, the complete relation atlas, and method boundaries.

The mathematical objects have now been unified.

All these definitions, relations, parameters, finite certificates, and quantifier interfaces can be carried further into a single executable interpreter.

This will make

theoretical language → executable relational language → repeatable computation and verification

three levels of one and the same structure.

Part VII RenSpectrum: Executable Implementation of the Theory and an Open Relational Language

— Fixed-Spectrum Interpreter, Formal-Proof Interface, Relation Compilation, Computational Engine, and Open Common Entry Point

Purpose of this part

The first six parts established the mathematical body of Unified Prime Relations Theory.

That body includes:

fixed identity spectrum → relation reconstruction → derived geometry and relation slices → CRT survival trees and genuine

This part converts these objects into executable form.

RenSpectrum is not the source of the theory's proofs; it is the executor, checker, and experimental front end of the theory.

Mathematical theorems determine what may be deduced; the formal proofs of Part VI determine how classical propositions enter the unified relational language without loss; RenSpectrum compiles definitions into finite tasks, expands relations into itemwise identities, presents certificates and computational paths in full, and preserves every finite verification as a repeatable record.

Hence:

classical proposition ↔ relational normal form → RSL → RenSpectrum execution → certificate and verification level.

The first bidirectional arrow belongs to mathematical formal proof; the subsequent arrows belong to executable implementation. They share the same relation object.

Abstract

RenSpectrum is the executable implementation of Unified Prime Relations Theory developed in this book.

Its Chinese short name is:

任谱

Its full Chinese name is:

任谱 · 统一素数关系工作台

Its English workbench name is:

RenSpectrum · Unified Prime-Relation Workbench

Its formal English name is:

Ren Prime-Relation Spectrum (RPRS):

A Fixed-Spectrum Interpreter for Prime and Arithmetic Relations

"Ren" corresponds to the Chinese character "任"; "Spectrum" refers to the fixed arithmetic identity spectrum and to the spectrum of relations read from that same fixed spectrum. RenSpectrum therefore does not mean "construct another table of primes." Rather, it means:

compile, read, and check distinct arithmetic relations on a fixed spectrum of integer identities.

The core principle of the software is identical to that of the book:

the matrix provides identities; relations select what is read; propositions specify quantifiers.

Part VI has already written Goldbach, Twin, Lemoine/Lévy, weak Goldbach, Chen, Polignac, Sophie Germain, prime k-tuples, Green–Tao, Dirichlet, Bertrand, Legendre, Oppermann, Brocard, Andrica, Firoozbakht, n^2+1, Bunyakovsky, Schinzel H, Mersenne, Grimm, Erdős–Woods, and other propositions as itemwise formal equivalence chains.

Accordingly, every executable object in RenSpectrum can carry complete formal-proof metadata:

classical proposition + relational normal form + quantifier type + certificate interface + verification level.

For example, the relational normal form of Goldbach is read as

$$G(m) > 0 \Leftrightarrow \Gamma_{\infty}^{(\sigma=m)} \neq \varnothing,$$

while the relational normal form of Twin is read as

$$C_T(X) \text{ unbounded} \Leftrightarrow \#\Gamma^{[1]} = \infty.$$

The software executes finite instances and certificates; formal proof guarantees that these executable objects have the same mathematical meaning as the classical propositions.

RenSpectrum realizes this principle through seven executable modules:

1. advance estimation of task size and device capability;
2. rapid relation readout;
3. interfaces for infinitude and universality;
4. seams, cycles, and sustainability;
5. atomwise / itemwise relation spectra;
6. local obstruction and emission–absorption certificates;
7. the open relational language RSL.

The rapid-readout module compiles built-in relations uniformly into

$$\mathcal{Q}_{\Theta,x}[\mathcal{B}_{\Theta}(\boldsymbol{I}(f_1(x;\Theta)),\dots,\boldsymbol{I}(f_r(x;\Theta)))].$$

The system simultaneously displays:

- the anchor;
- the position maps;
- the fixed identity at each position;
- the local relation symbol;
- hit / non-hit status in the current finite horizon;
- the quantifier type;
- applicable auxiliary certificates.

The infinitude module does not merge all infinite propositions into one category. It recognizes

INF_RECURRENCE, UNIVERSAL_FIBER, WINDOW_UNIVERSAL,

ARBITRARILY_LONG, PARAMETER_INFINITE, UNIVERSAL_PREDICATE, BLOCK_SEPARATION.

The seam module simultaneously reads

$$\boxed{\text{generation seam } s^2,}$$

$$\boxed{\text{phase seam } W_z,}$$

and the family of all prime-square seams:

$$\boxed{\text{square-seam family.}}$$

A fixed-offset pattern

$$H$$

is automatically passed to the local-obstruction and emission–absorption modules. The system computes

$$v_H(q), \quad v_H(q), \quad B_H, \quad U, \quad \mu_H(Q),$$

$$C_H^{\text{exact}}, \quad C_H^{\text{struct}}, \quad e_H^{\text{exact}}, \quad e_H^{\text{struct}}, \quad T_H, \quad D_H.$$

These satisfy

$$\boxed{D_H \le \mu_H(Q)|U|,}$$

and

$$\boxed{T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}.}$$

Built-in relation families include fixed sums, fixed differences, consecutive fixed gaps, affine relations, finite patterns, arithmetic-progression families, square windows, neighboring-prime indices, exponential / polynomial pullbacks, complete factor-set relations, and several mixed-relation examples.

The preset catalog is not the theoretical boundary of the system. The open **Relation Specification Language (RSL)** allows users to define new relations through

$$\boxed{\text{anchor} + \text{parameters} + \text{maps / map_generator} + \text{predicate} + \text{quantifier}}$$

and optionally connect them to balance / emission–absorption certificates.

The fundamental discipline of RSL is

if a new relation can be written as a new readout, do not add a new matrix mechanism.

Only when the existing readout primitives cannot express a relation is the relational language extended; such extensions may increase readout capability but must not rewrite the fixed identity spectrum.

The computational layer begins by estimating the task budget. The current Web edition classifies tasks as

LIGHT | MODERATE | HEAVY / LOCAL RECOMMENDED | BLOCKED ON WEB.

The dense-spectrum backend mainly uses

spf:Int32Array

and

prime:Uint8Array,

with raw-array memory approximately

$$M_{\mathrm{raw}}(N) \approx 5(N+1) \text{ bytes.}$$

Engineering hard thresholds for the Web edition include

$$N_{\mathrm{dense}}^{\mathrm{hard}} = 3 \times 10^7,$$

$$A_{\mathrm{hard}} = 2 \times 10^6,$$

$$A_{\mathrm{heavy}} = 5 \times 10^5,$$

$$R_{\mathrm{DOM}} = 5000,$$

and, for the browser version of the Mersenne Lucas–Lehmer test,

$$p \leq 5000.$$

The exact-integer boundary for ordinary JavaScript integers is

$$2^{53} - 1 = 9\,007\,199\,254\,740\,991.$$

Integer identities, divisibility, and modular arithmetic beyond this boundary must enter

BigInt

or a local arbitrary-precision integer engine.

The high-performance implementation uses a layered backend:

Web JS → Web Worker / Wasm → Local CPU → Local CPU + GPU.

The local edition no longer uses one monolithic dense array from 0 to N. Instead, it adopts

segmented sieve + segment streaming.

Therefore

coordinate magnitude ≠ window width.

A narrow window near 10^{18} does not require construction of the complete spectrum on $[0, 10^{18}]$.

This part also establishes verification levels for RenSpectrum so that machine output corresponds one-to-one with mathematical levels:

$$\mathrm{V}_0 \to \mathrm{V}_1 \to \mathrm{V}_2 \to \mathrm{V}_3 \to \mathrm{V}_4 \to \mathrm{V}_5.$$

They mean, respectively:

- syntax is executable;
- identity readout is independently checkable;
- finite relation readout is independently checkable;
- a finite terminal certificate holds;
- a structural batch certificate holds;
- global closure matched to the target quantifier holds.

Verification levels are not software disclaimers. They directly encode levels of mathematical objects.

Finally, this part gives a complete parameter dictionary and reproducibility protocol, so that every relation readout can preserve

relation definition + parameters + finite horizon + control depth + backend + software version + certificate + output summ

RenSpectrum thereby becomes an open relation interpreter.

It does not package prime-number problems into a small set of buttons; it hands the relational language developed throughout the book to the reader for continued use.

Keywords: RenSpectrum; 任谱; RPRS; fixed-spectrum interpreter; formal-proof interface; RSL; relation compilation; infinitude interface; seam; emission–absorption; BigInt; segmented sieve; WebAssembly; WebGPU; verification level; reproducible experiment; open common entry point

Part I Name, Positioning, and Execution Boundaries

Chapter 1 RenSpectrum: The Full Meaning of the Name

The executable system of this book is named

RenSpectrum

Chinese short name:

任谱

Full Chinese name:

任谱 · 统一素数关系工作台

English workbench name:

RenSpectrum · Unified Prime-Relation Workbench

Formal English name:

Ren Prime-Relation Spectrum (RPRS):

A Fixed-Spectrum Interpreter for Prime and Arithmetic Relations

The name has three layers of meaning.

1.1 Ren

“Ren” corresponds both to the Chinese surname “任” and to the “任” in “任谱”.

It gives the software an unambiguous identity while retaining a natural correspondence with the Chinese name.

1.2 Spectrum

“Spectrum” does not mean that prime numbers are being interpreted as a physical frequency spectrum.

Here it refers to

the fixed arithmetic identity spectrum

and to

the spectrum of relations derived from that same fixed identity spectrum.

The identity at every integer position is fixed;

different relations change only

$$f_i, \quad \mathcal{B}, \quad \mathcal{Q}.$$

Therefore

the spectrum is fixed; the readouts are open.

(1.1)

1.3 Prime-Relation

"Prime-Relation" emphasizes that RenSpectrum is not merely a prime table.

Its basic object is not the single sequence

$$2,3,5,7,\ldots$$

but rather

$$\boxed{\text{positional identity} + \text{relations among positions.}}$$

Its fundamental unit is therefore

$$\mathfrak{R}_{\Theta,x},$$

not a single prime flag.

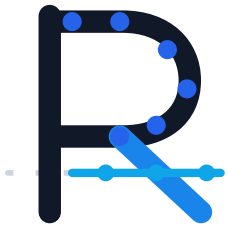
1.4 Workbench

"Workbench" means that the system is designed as a working environment in which one can:

- enter a relation;
- compile the relation;
- read the fixed spectrum;
- expand itemwise instances;
- check local obstruction;
- compute finite certificates;
- compare quantifier types;
- reproduce experiments.

Thus it is neither a static database nor a calculator with only one operation.

Chapter 2 The Visual Identity Is Also Part of the Relational Language



The RenSpectrum Web identity uses an abstract R-shaped skeleton overlaid with discrete spectral points and horizontal relation lines.

Its visual grammar can be read as

$$\boxed{\text{R-shaped body} + \text{discrete spectral points} + \text{relation links.}}$$

These correspond to three levels:

- the fixed identity spectrum;
- relation direction;
- relation readout.

The website provides both

$$\boxed{\text{中文 / English}}$$

and

$$\boxed{\text{light / dark / system}}$$

theme choices.

Themes change only visual presentation; they do not alter any relation definition, parameter, computation, or certificate.

Therefore

the representation layer may change; the mathematical semantics remain invariant.

(2.1)

Chapter 3 The Software Is Not the Source of Proof, but an Execution Environment for Proof Semantics

A mathematical proposition is established by

$$\boxed{\text{definitions} + \text{theorems} + \text{proofs.}}$$

Software executes

$$\boxed{\text{finite input} \rightarrow \text{finite computation} \rightarrow \text{finite output.}}$$

The correct relation between them is

$$\boxed{\text{the theory defines the computational object} \rightarrow \text{the software executes the object} \rightarrow \text{the output is i}} \tag{3.1}$$

For example,

$$e_H^{\text{struct}} > 0$$

implies

$$T_H \geq 1$$

not because the webpage displays a green status, but because Part V has already proved

$$T_H \geq e_H^{\text{struct}}.$$

The webpage merely computes and displays this certificate.

Thus

a program can verify an already defined finite certificate; the mathematical force of that certificate comes from its theorem.

(3.2)

Chapter 4 The Seven Execution Layers of RenSpectrum

The Web workbench is divided into seven principal regions:

0.	task scale / device-capability precheck
1.	rapid relation readout
2.	infinitude / universality interface
3.	seams / cycles / sustainability
4.	atomwise / itemwise relation spectrum
5.	local obstruction + emission–absorption
6.	master table of built-in relation types
7.	advanced RSL extensions.

(4.1)

The label 0 means:

> Before computation begins, determine what the task is, how many resources it requires, and which backend should execute it.

This is consistent with the discipline of the book: first determine the object, then choose the method.

Figure 7-01 The Seven Execution Layers of RenSpectrum

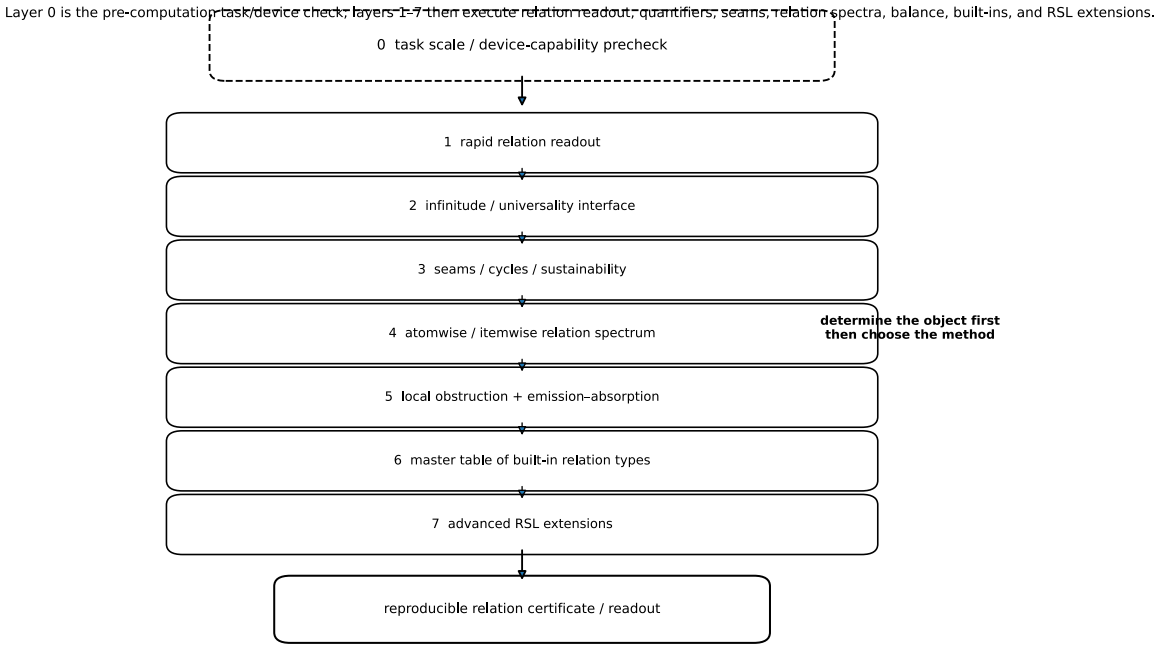


Figure 7-01. The Seven Execution Layers of RenSpectrum. Layer 0 is the pre-computation task-scale and device-capability check. Layers 1–7 then handle rapid relation readout, infinitude/universality, seams and sustainability, itemwise relation spectra, local obstruction and emission–absorption, built-in relation types, and advanced RSL extensions.

Part II Rapid Relation Readout

Chapter 5 Unified Compilation Formula for Rapid Readout

Every built-in relation is first compiled into

$$\boxed{R_\Theta(x) = \mathcal{B}_\Theta(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_r(x;\Theta)))}. \tag{5.1}$$

It is then supplemented by

$$\boxed{\mathcal{Q}_{\Theta,x}\cdot}$$

The "relational normal form" displayed by the workbench is the visualization of this structure.

Thus rapid readout is not implemented by writing unrelated code for every conjecture.

All built-in relations enter through the same compilation path:

$$\boxed{\text{relation type} \to \text{parameters} \to \text{position maps} \to \text{identity readout} \to \text{local predicate} \to \text{quantifier.}} \tag{5.2}$$

Chapter 6 Anchor: The Free Position of a Relation Instance

RenSpectrum uses

anchor

to denote the free variable of a relation instance.

For example:

Goldbach

$$x = p.$$

Twin

$$x = p.$$

Prime tuple

$$x = n.$$

Fixed Green–Tao slice

$$x = a.$$

Bunyakovsky

$$x = n.$$

The anchor is not the relation itself.

It is only

the free coordinate selecting a concrete relation instance under fixed parameters.

Chapter 7 Input Fields for Rapid Readout

Rapid relation readout accepts the following inputs in a unified manner.

7.1 Relation Type

This determines

$$f_i, \quad \mathcal{B}, \quad \mathcal{Q}.$$

7.2 Parameter 1 / Parameter 2 / Parameter 3

Different relations use different parameters.

For example:

- Goldbach: N;
- fixed gap: d;
- affine relation: A, B;
- Green–Tao: k, d;
- Erdős–Woods: x, y, k.

7.3 Readout Horizon

The readout horizon restricts the current finite execution region.

It does not change the global proposition; it specifies only how many instances the current executor actually expands.

7.4 Control Depth z

Two modes are provided:

$$\boxed{\text{automatic: } \sqrt{\text{current largest relation coordinate}}}$$

or

$$\boxed{\text{manual } z.}$$

Automatic mode is used for complete finite primality certification;

manual mode is used to observe intermediate structure under a selected finite control layer.

7.5 Display Filtering

Itemwise results may be displayed as

$$\boxed{\text{all} \mid \text{hits only} \mid \text{non-hits only.}}$$

Filtering changes only the display, not the internal statistics.

Figure 7-02 Unified Compilation Pipeline for Rapid Readout

All built-in relations share one compilation entry: relation type and parameters become position maps, then pass through fixed-identity readout, a local predicate, and a quantifier module.

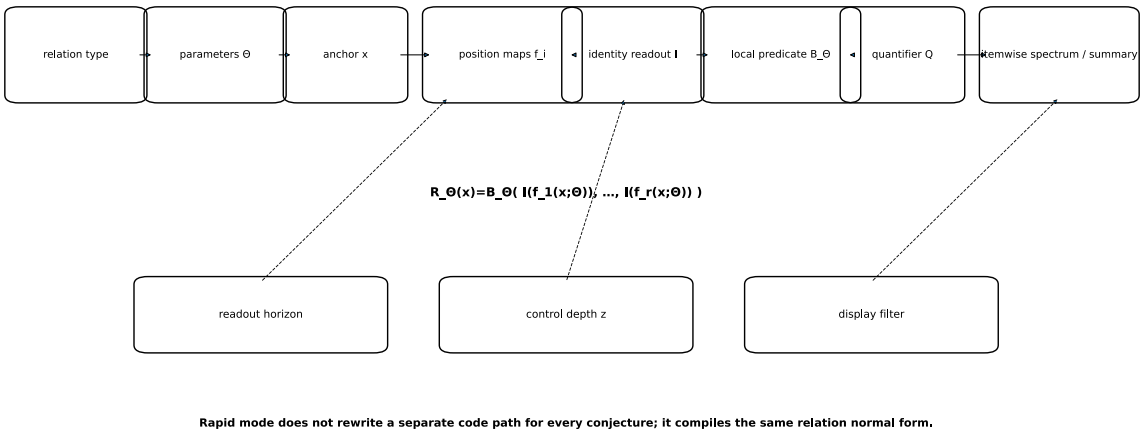


Figure 7-02. Unified Compilation Pipeline for Rapid Readout. RenSpectrum compiles a built-in relation uniformly as $R_\Theta(x) = \mathcal{B}_\Theta(\boldsymbol{I}(f_1(x;\Theta)), \dots, \boldsymbol{I}(f_r(x;\Theta)))$, then attaches the outer quantifier $\mathcal{Q}_{\Theta,x}$. The execution input consists of relation type, parameters, anchor, position maps, fixed identity, local predicate, and quantifier; the readout horizon, control depth z, and display filter govern finite execution without changing the global proposition.

Chapter 8 The Itemwise Relation Spectrum

A single rapid readout may produce many relation instances.

Each row preserves at least

$$\boxed{\text{anchor} + \text{position / mirror} + \text{identity} + \text{local information} + \text{result.}} \tag{8.1}$$

Identity notation uses

$$\boxed{A}$$

for an atom,

$$\boxed{\mathcal{M}_q}$$

for the multiplicative identity whose least prime factor is q, and

$$\boxed{\partial}$$

when a relation position does not lie in the valid positive-integer identity domain.

Thus the itemwise spectrum does not merely display

HIT / MISS.

It also preserves the reason for failure.

Chapter 9 Fixed Sums and Weighted Reflection

The built-in Goldbach relation is

$$\boxed{p \mapsto N - p.}$$

Its local readout is

$$\alpha(p)\alpha(N - p).$$

Its quantifier is

$$\text{UNIVERSAL_FIBER.}$$

Lemoine / Lévy uses

$$\boxed{q = \frac{N - p}{2}.}$$

This is a weighted reflection.

Both belong to parameterwise target fibers, but their position maps differ.

Chapter 10 Fixed Differences and Consecutive Fixed Gaps

Twin uses

$$\boxed{p \mapsto p + 2.}$$

A general fixed difference uses

$$\boxed{p \mapsto p + d.}$$

Both require only that the two endpoints be atoms.

A Polignac-type consecutive prime gap additionally requires

$$\boxed{(p, p + d) \text{ contain no other prime in their open interval.}}$$

Accordingly, the workbench strictly separates

$$\boxed{\text{fixed difference}}$$

from

$$\boxed{\text{consecutive fixed gap.}}$$

Chapter 11 Affine Relations

Sophie Germain uses

$$\boxed{p \mapsto 2p + 1.}$$

A general affine relation uses

$$\boxed{p \mapsto Ap + B.}$$

The workbench does not need a new relation type for every choice of A, B.

They all share the same affine reader.

Chapter 12 Fixed-Offset Patterns and a Blocking Example

A prime tuple uses

$$\boxed{n \mapsto n + H.}$$

For example,

$$H = \{0,2,6,8\}.$$

The workbench also includes

$$\boxed{H = \{0,2,4\}}$$

as an example of complete local obstruction.

Modulo 3, the residues

$$0,2,1$$

cover all residue classes.

Thus the long-term all-atomic recurrence of this relation is closed already at the local-obstruction layer.

This example is also an important benchmark in the system’s regression tests.

Chapter 13 Fixed Green–Tao Slices

Given

$$k, \quad d,$$

generate

$$\boxed{H(k,d) = \{0, d, \ldots, (k-1)d\}.}$$

The current workbench reads a fixed

$$(k, d)$$

slice.

Its global quantifier type is displayed as

$$\boxed{\text{ARBITRARILY_LONG.}}$$

This reminds the user that

> a hit in the current finite slice and the statement “arbitrarily long” belong to two different levels.

Chapter 14 Window Relations

14.1 Legendre

Window:

$$(n^2, (n+1)^2).$$

Readout:

does the window contain at least one atom?

14.2 Oppermann

Two windows:

$$(n(n-1), n^2),$$

$$(n^2, n(n+1)).$$

Both are required to be nonempty.

14.3 Brocard

For consecutive primes

$$p_n, p_{n+1},$$

read

$$\pi(p_{n+1}^2) - \pi(p_n^2).$$

The local predicate is

$$\geq 4.$$

Together these relations show that

a window is a first-class relation object.

Chapter 15 Neighboring-Prime Indices

Andrica reads

$$\boxed{\sqrt{p_{n+1}} - \sqrt{p_n} < 1.}$$

Firoozbakht is implemented in the numerically more stable logarithmic form

$$\boxed{\frac{\log p_{n+1}}{n+1} < \frac{\log p_n}{n}.}$$

Their anchor is the prime index n, not the integer position itself.

Chapter 16 Exponential and Polynomial Pullbacks

16.1 Mersenne

$$\boxed{p \mapsto 2^p - 1.}$$

Read through BigInt and a Lucas–Lehmer-type primality path.

16.2 $n^2 + 1$

$$\boxed{n \mapsto n^2 + 1.}$$

16.3 Bunyakovsky

Enter a polynomial expression

$$\boxed{f(n).}$$

The workbench reads

$$\alpha(f(n)).$$

16.4 Schinzel H

Enter a polynomial family

$$f_1(n), \dots, f_k(n).$$

Read

$$\prod_i \alpha\,(f_i(n)).$$

A local-obstruction check for the polynomial family may also be attached.

Chapter 17 Relations Using Complete Factor Identities

17.1 Grimm

For a consecutive block of composite integers, read

$$\mathrm{PF}(n)$$

at each position, then construct a matching among prime factors.

This requires

$$\boxed{\text{factor-set engine} + \text{matching engine.}}$$

17.2 Erdős–Woods

Compare two integer blocks

$$x, x+1, \ldots, x+k,$$

$$y, y+1, \ldots, y+k$$

and ask whether their

$$\mathrm{PF}$$

values agree at every position.

Quantifier interface:

$$\boxed{\mathrm{BLOCK_SEPARATION.}}$$

Thus the fixed spectrum of RenSpectrum is not merely a single prime bit.

It can also read complete factor identities.

Chapter 18 Three Examples of Mixed Relations

The workbench retains three open examples to show that the relational language can freely compose existing readout primitives.

18.1 Cross-Mirror Relation

$$\boxed{p, \quad p+2, \quad 2p+1}$$

are all atoms.

This mixes:

- translation;
- affine mapping.

18.2 Quadratic Double-Atom Relation

$$\boxed{n^2+n+41, \quad n^2+n+43}$$

are both atoms.

This is a double polynomial pullback.

18.3 Reflection–Translation Coupling

$$\boxed{p, \quad p+2, \quad N-p}$$

are all atoms.

This mixes:

- fixed difference;
- fixed-sum reflection.

None of these three examples requires a new integer matrix.

Only existing position maps need to be composed.

Figure 7-03 Data Model: anchor / maps / predicate / quantifier

Goldbach, Twin, and prime tuples use the same four-field model; what changes is the concrete position maps and quantifier.

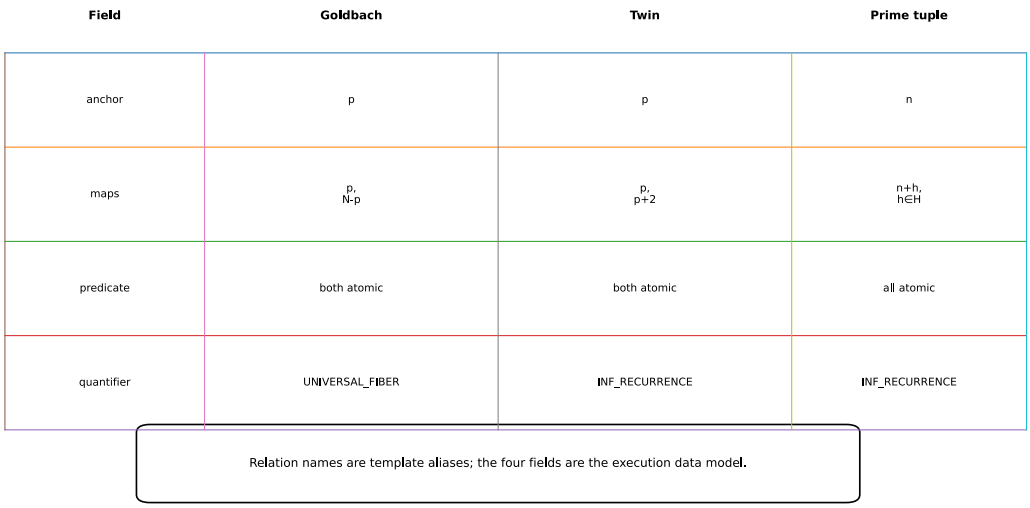


Figure 7-03. Data Model: anchor / maps / predicate / quantifier. The core data model of rapid relation readout compresses to four field classes: anchor gives the free position, maps generate the concrete relation endpoints, predicate checks their fixed identities and any additional local conditions, and quantifier specifies the global readout. Goldbach, Twin, and prime tuples therefore share the same execution structure and differ only in position maps, local predicates, or quantifier parameters.

Part III Infinitude and Universality Modules

Chapter 19 The Software First Identifies the Quantifier Type

RenSpectrum does not mistake a finite hit count for a global conclusion.
It first identifies the quantifier type of the relation, and only then displays the matching criterion.
This corresponds to the quantifier structure established in Part V.

Chapter 20 INF_RECURRENCE

Infinite recurrence uses

$$\boxed{C(X) = \sum_{x \le X} R\,(x).}$$

The global criterion is

$$\boxed{C(X) \text{ is unbounded.}}$$

The current Web page displays only

$$\boxed{C_{\text{current}},}$$

a finite section of the cumulative sequence.

This applies to:

- Twin;
- fixed gap;
- Sophie Germain;
- prime tuple;
- $n^2 + 1$;
- Bunyakovsky;
- Schinzel H;
- Mersenne.

Chapter 21 UNIVERSAL_FIBER

Form:

$$\boxed{\forall\theta\ \exists x.}$$

The current page displays the hit count of one fixed parameter fiber:

$$\boxed{C(\theta).}$$

The global criterion is

$$\boxed{C(\theta) > 0 \quad \forall\theta.}$$

This applies to:

- Goldbach;
- Lemoine / Lévy;
- reflection–translation coupled universal fibers.

Chapter 22 WINDOW_UNIVERSAL

The current finite readout displays

$$\boxed{\text{number of nonempty windows/total number of windows.}}$$

The global criterion is

$$\boxed{W(n) \geq 1 \quad \forall n.}$$

A typical example is:

- Legendre.

Chapter 23 UNIVERSAL_PREDICATE

Form:

$$\boxed{\forall n, \quad P(n).}$$

The current workbench displays

$$\boxed{\text{number passed/total number read.}}$$

Typical examples:

- Oppermann;
- Brocard;
- Andrica;
- Firoozbakht;
- Grimm.

Chapter 24 ARBITRARILY_LONG

Form:

$$\boxed{\forall k\ \exists \text{ a relation of length } k.}$$

The global criterion is

$$\boxed{L(X) \text{ is unbounded.}}$$

The current fixed Green–Tao slice reads only fixed

$$(k, d).$$

The workbench therefore displays separately

$$\boxed{\text{hit in the fixed slice}}$$

and

$$\boxed{\text{unbounded maximum realized length.}}$$

Chapter 25 PARAMETER_INFINITE

This interface tracks

the number of successful parameter fibers.

If

$$P(X)$$

is unbounded, then infinitely many parameters succeed.

It applies to problems in which

the parameter itself is required to occur infinitely often, while each successful parameter requires only a finite witness.

Chapter 26 BLOCK_SEPARATION

For two factor-identity blocks

$$B_K(x), \quad B_K(y),$$

the current readout displays:

- whether they remain completely identical;
- the length at which they have already separated.

The global target is to find a uniform finite

$$\boxed{K_*}.$$

This interface is entirely different from infinite recurrence.

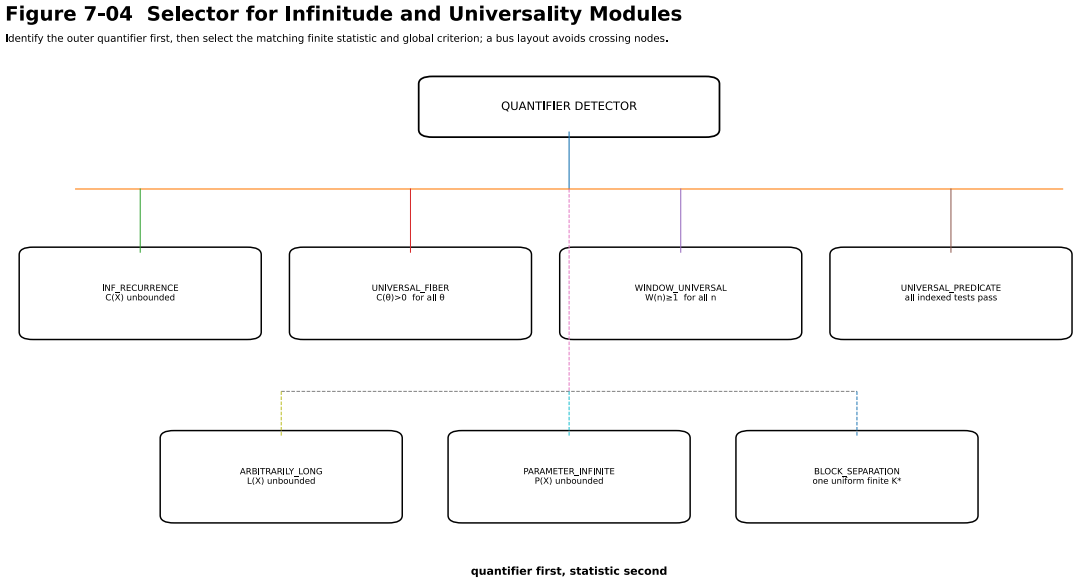


Figure 7-04. Selector for Infinitude and Universality Modules. RenSpectrum first identifies the outer quantifier and then selects the matching finite statistic and global criterion: INF_RECURRENCE reads unbounded $C(X)$; UNIVERSAL_FIBER requires $C(\theta) > 0$ for every parameter; WINDOW_UNIVERSAL requires $W(n) \geq 1$ for every window; UNIVERSAL_PREDICATE requires every indexed predicate to pass; ARBITRARILY_LONG and PARAMETER_INFINITE read unbounded $L(X)$ and $P(X)$ respectively; BLOCK_SEPARATION searches for one uniform finite separation length K_*.

Part IV Seams, Cycles, and Persistence

Chapter 27 Executable Definitions of Three Types of Seams

The Web page displays three kinds of seams simultaneously.

27.1 Generation Seam

$$\boxed{s^2.}$$

This is the first location at which the next control atom s can enter a new least-prime-factor decision.

It answers

$$\boxed{\text{when does the control layer turn over?}}$$

27.2 Phase Seam

Fix the control depth z, and let

$$\boxed{W_z = \prod_{\substack{q \le z \\ q \text{ odd prime}}} q\,.}$$

This is the return point of the common residue phase of the finite control layer.

It answers

when does the system return to the same phase?

27.3 Square-Seam Family

All prime squares

$$\boxed{r^2}$$

form a boundary family for relation windows.

They are used for:

- relation segmentation;
- cross-seam statistics;
- persistence scans;
- square-window certificates.

Chapter 28 Finite Cycle S_0 and Global Persistence S_∞

When the set of control primes is frozen, the finite modular phase returns exactly.
Denote this by

$$\boxed{S_0.}$$

When the control layer continues to grow, the modulus itself also grows.
The global problem then enters

$$\boxed{S_\infty.}$$

Hence

a finite phase cycle is not the same thing as an infinite relation cycle.

(28.1)

RenSpectrum displays them separately.

Chapter 29 Square-Seam Closure Scan

A seam scan records at least

$$\boxed{\text{control prime} + r^2 + \text{number of successful relations crossing the seam} + \text{decision.}} \tag{29.1}$$

Its purpose is not to turn “crossing a seam” into an independent new proposition.

It provides

$$\boxed{\text{a persistent history of the relation across consecutive finite windows.}}$$

Chapter 30 Persistent Readout Across Square Windows

For every square window, the system records

$$\boxed{\text{window} + \text{hit count} + \text{status.}}$$

This sequence corresponds directly to the persistent non-extinction relation sieve of Part V. When a relation belongs to

$$\text{INF_RECURRENCE},$$

the genuine global criterion is whether the cumulative hit count is unbounded.

Part V Local Obstruction and the Emission–Absorption Executor

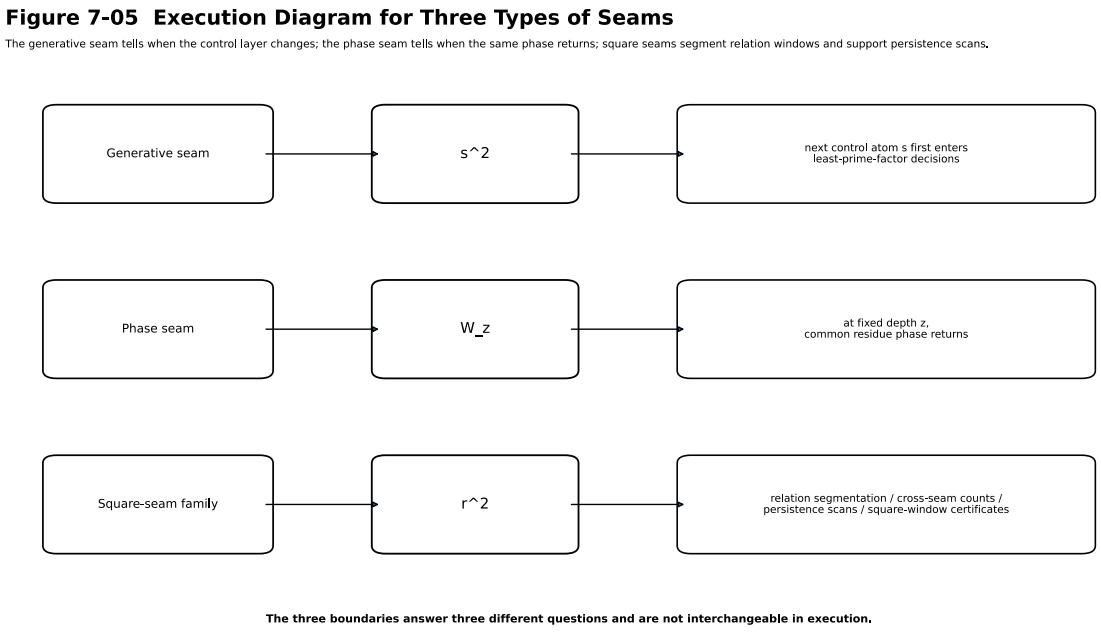


Figure 7-05. Execution Diagram for Three Types of Seams. RenSpectrum distinguishes three seams simultaneously. The generative seam s^2 marks the first point at which the next control atom s can enter a new least-prime-factor decision. The phase seam $W_z = \prod_{q \le z,\, q \text{ odd prime}} q$ is the return period of the common residue phase at fixed control depth. The square-seam family r^2 is used for relation segmentation, cross-seam statistics, persistence scans, and square-window certificates. The three seams answer different questions and are not interchangeable.

Chapter 31 Normalization of a Pattern H

When a relation can be written as a fixed-offset pattern

$$\boxed{H = \{h_1, \dots, h_k\},}$$

the system automatically activates the pattern module.

It first normalizes the pattern by:

1. removing duplicates;
2. sorting;
3. translating the whole pattern so that the smallest offset is 0.

Thus

$$\boxed{H \sim H - c.}$$

A global translation does not change the internal difference structure of the pattern.

Chapter 32 Local-Obstruction Fingerprint

For each prime q, compute

$$\boxed{\nu_H(q) = \#\{h \bmod q : h \in H\}.}$$

and

$$\boxed{\nu_H(q) = q - \nu_H(q).}$$

If

$$\nu_H(q) = q,$$

then the pattern is completely covered locally by q.

The Web page immediately displays

$$\boxed{\text{NO INFINITE RECURRENCE}}$$

as the pattern-level local conclusion and stops meaningless $B/U/e$ calculations.

This is not a heuristic stopping rule.

It corresponds to the local-obstruction theorem of Part IV.

Chapter 33 Admissible Status of a Pattern

If there is no complete local cover, the Web page displays

ADMISSIBLE.

Its strict meaning is

the local spectrum permits long-term all-atomic relations.

It does not mean

infinitely many all-atomic relations have already been generated.

An admissible pattern therefore continues into the supply–absorption module.

Chapter 34 B_H: Relation Emission

In the square horizon

$$(r^2, s^2),$$

define

$$\boxed{B_H(r,s)}$$

to be the number of complete pattern instances surviving all old historical layers.

"Emission" means

how many complete candidate relations the old relation matrix supplies in the current finite horizon.

It is not a count of single-point survivors.

Chapter 35 U: Absorption Resources

Define

$$\boxed{U(r,s) = \left\{u \in \mathbb{P} : s \leq u < \frac{s^2}{r}\right\}.}$$

It records the finite cofactor resources through which the current last control prime r can produce genuine deaths via

$$ru.$$

Chapter 36 Q and $\mu_H(Q)$

The user may choose a finite wheel

$$Q.$$

Define

$$M_Q = \prod_{q \in Q} q.$$

The system enumerates unit residue classes

$$c \in (\mathbb{Z}/M_Q\mathbb{Z})^\times$$

and computes

$$J_H(c; Q).$$

The maximum

$$\boxed{\mu_H(Q) = \max_c |J_H(c; Q)|}$$

means

the maximum number of pattern roles for which a single resource can be reused as an absorber.

Chapter 37 Two Absorption Capacities

Fixed-spectrum resource capacity:

$$\boxed{C_H^{\text{exact}} = \mu_H(Q)|U|.}$$

Structural capacity:

$$\boxed{C_H^{\text{struct}} = \mu_H(Q)(\Delta - 1),}$$

where

$$\Delta = s - r.$$

The former uses the number of resources actually present in the current fixed spectrum; the latter uses only the structural upper bound.

Chapter 38 Two Net Relation Potentials

Define

$$\boxed{e_H^{\text{exact}} = B_H - C_H^{\text{exact}},}$$

$$\boxed{e_H^{\text{struct}} = B_H - C_H^{\text{struct}}.}$$

By the theorem of Part V,

$$\boxed{T_H \geq e_H^{\text{exact}} \geq e_H^{\text{struct}}.}$$

The Web page can therefore mark, window by window:

- a fixed-spectrum resource certificate;
- a structural certificate.

Chapter 39 T_H and D_H

The number of genuine all-atomic relations is

$$\boxed{T_H.}$$

The number of genuine deaths is

$$\boxed{D_H = B_H - T_H.}$$

Thus every row preserves the complete ledger

$$\boxed{p,\ r,\ s,\ \Delta,\ B_H,\ U,\ \mu_H, C_{\text{exact}}, C_{\text{struct}}, e_{\text{exact}}, e_{\text{struct}}, T_H, D_H.} \tag{39.1}$$

This is the executable form of the supply–absorption ledger of Part V.

Chapter 40 Which Relations Enter the Balance Module Automatically?

Fixed-offset structures can connect directly to this module:

- Twin;
- fixed gap;
- the endpoint pattern of Polignac;
- prime tuple;
- fixed (k, d) slices of Green–Tao;
- fixed-offset relations defined by RSL.

Relations without a fixed-offset pattern display

not applicable.

“Not applicable” is not a software defect; it means that the relation requires a different finite-control geometry.

Figure 7-06 Supply–Absorption Executor

A fixed-offset pattern is normalized and checked for local admissibility, then each square horizon computes supply, resources, reuse, capacity, net potential, and the genuine/dead relation ledger.

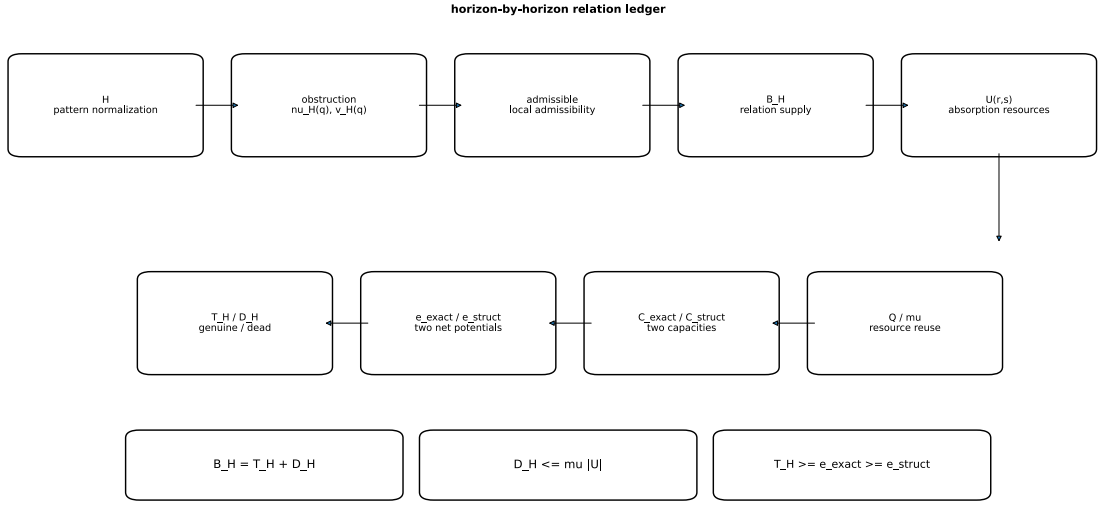


Figure 7-06. Supply–Absorption Executor. Once a fixed-offset pattern enters the balance module, the software records the pattern H, its local-obstruction fingerprint and admissible state, relation supply B_H in the square horizon, absorption resources $U(r, s)$, wheel Q and reuse degree $\mu_H(Q)$, exact/structural capacities, exact/structural net potentials, and the genuine/dead counts T_H, D_H. Every horizon must also satisfy $B_H = T_H + D_H$, $D_H \le \mu_H(Q)|U|$, and $T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}$.

Part VI RSL: The Open Relational Language

Chapter 41 Name and Purpose of RSL

RSL stands for

Relation Specification Language

—that is,

a language for specifying relations.

It turns the relational normal forms established in Part III into executable descriptions.

The purpose of RSL is not to create new arithmetic identities.

It describes only

where to read + how to combine + which quantifier interprets the result.

Therefore

the preset catalog is not the boundary of the theory; RSL is the relation interface.

(41.1)

Chapter 42 The Matrix–Subrelation Discipline of RSL

RSL must obey

the matrix determines the subrelation readout; the subrelation may not modify the matrix in reverse.

(42.1)

A new relation is therefore first written, if possible, as

$$\boxed{\text{maps / map_generator + predicate + quantifier.}}$$

If it can be expressed this way,

$$\boxed{\text{no new matrix mechanism is introduced.}} \tag{42.2}$$

Only when the readout primitives themselves cannot express a relation is RSL extended.

Such an extension may add

$$\boxed{\text{readout capability}}$$

but it may not alter

$$\boxed{\boldsymbol{I}(n).}$$

Chapter 43 Basic Structure of RSL

An RSL object contains:

```
{
  "name": "...",
  "anchor": {
    "var": "...",
    "domain": "...",
    "start": 0,
    "end": 100
  },
  "parameters": {},
  "maps": [],
  "predicate": {},
  "quantifier": {}
}
```

or uses

```
{
  "map_generator": {}
}
```

to generate a position bundle of variable length.

Chapter 44 The anchor Field

An anchor contains at least

var,	domain,	start,	end.

The domain may be

integer,	prime,	odd,	even.

For example:

```
"anchor": {
  "var": "p",
  "domain": "prime",
  "start": 2,
  "end": 1000
}
```

Chapter 45 The `parameters` Field

Parameters are external constants within a fixed relation family.

For Twin:

```
"parameters": {
  "d": 2
}
```

For Goldbach:

```
"parameters": {
  "N": 100
}
```

For a fixed Green–Tao slice:

```
"parameters": {
  "k": 5,
  "d": 30
}
```

Chapter 46 maps

A fixed-length relation uses:

```
"maps": [
  {"id": "x0", "expr": "p"},
  {"id": "x1", "expr": "p+d"}
]
```

Mathematically,

$$\boxed{x_i = f_i(\text{anchor}; \Theta).}$$

Expressions support integer-arithmetic combinations such as

$$+, -, *, /, \cdot$$

Every relation position must ultimately lie in a readable integer-identity domain.

Chapter 47 `map_generator`

A relation whose length is parameter-dependent uses:

```
"map_generator": {
  "id_prefix": "x",
  "index": "j",
  "from": 0,
  "to": "k-1",
  "expr": "a+j*d"
}
```

It generates

$$a,\ a+d,\ \dots,a+(k-1)d.$$

Thus a Green–Tao-type relation does not require manually writing

$$x_0, x_1, \dots, x_{k-1}.$$

Chapter 48 predicate

The local predicate determines how these positions are combined.

Basic predicates include

all_prime

is_prime

any_prime

no_prime_between

and logical operators

and, or, not.

Generated bundles may also use

all_generated_prime.

Chapter 49 quantifier

The quantifier field explicitly distinguishes

INF_RECURRENCE,

UNIVERSAL_FIBER,

WINDOW_UNIVERSAL,

ARBITRARILY_LONG,

PARAMETER_INFINITE,

UNIVERSAL_PREDICATE,

BLOCK_SEPARATION.

Thus RSL does not merely describe local relations.
It separates

relation

from

proposition

as two distinct levels.

Chapter 50 RSL Expression of Twin

```
{
  "name": "Twin",
  "anchor": {
    "var": "p",
    "domain": "prime",
    "start": 2,
    "end": 1000
  },
  "parameters": {
    "d": 2
  },
  "maps": [
    {"id": "x0", "expr": "p"},
    {"id": "x1", "expr": "p+d"}
  ],
  "predicate": {
    "op": "all_prime",
    "args": ["x0", "x1"]
  },
  "quantifier": {
    "type": "INF_RECURRENCE"
  }
}
```

This specification corresponds exactly to

$$p,\ p+2 \in \mathbb{P},$$

and to

$$\exists^{\infty} p.$$

Chapter 51 RSL Expression of Goldbach

```
{
  "name": "Goldbach N=100",
  "anchor": {
    "var": "p",
    "domain": "prime",
    "start": 2,
    "end": 50
  },
  "parameters": {
    "N": 100
  },
  "maps": [
    {"id": "x0", "expr": "p"},
    {"id": "x1", "expr": "N-p"}
  ],
  "predicate": {
    "op": "all_prime",
    "args": ["x0", "x1"]
  },
  "quantifier": {
    "type": "UNIVERSAL_FIBER"
  }
}
```

The current RSL object describes the single fiber

$$N = 100.$$

The global proposition is interpreted at the quantifier layer as

$$\forall N.$$

Chapter 52 RSL Expression of Polignac

```
{
  "name": "Polignac d=6",
  "anchor": {
    "var": "p",
    "domain": "prime",
    "start": 2,
    "end": 1500
  },
  "parameters": {
    "d": 6
  },
  "maps": [
    {"id": "x0", "expr": "p"},
    {"id": "x1", "expr": "p+d"}
  ],
  "predicate": {
    "op": "and",
    "args": [
      {
        "op": "all_prime",
        "args": ["x0", "x1"]
      },
      {
        "op": "no_prime_between",
        "lo": "x0",
        "hi": "x1"
      }
    ]
  },
  "quantifier": {
    "type": "INF_RECURRENCE"
  }
}
```

This shows that RSL can compose

endpoint identity + empty-interval readout.

Chapter 53 RSL Expression of a Fixed Green–Tao Slice

```
{
  "name": "Green-Tao fixed slice",
  "anchor": {
    "var": "a",
    "domain": "integer",
    "start": 2,
    "end": 3000
  },
  "parameters": {
    "k": 5,
    "d": 30
  },
  "map_generator": {
    "id_prefix": "x",
    "index": "j",
    "from": 0,
    "to": "k-1",
    "expr": "a+j*d"
  },
  "predicate": {
    "op": "all_generated_prime"
  },
  "quantifier": {
    "type": "ARBITRARILY_LONG"
  }
}
```

This shows that

> a variable-length relation is an object of the language layer, not a new matrix object.

Chapter 54 A Mixed Polynomial Relation

An open example:

```
{
  "name": "Mixed polynomial",
  "anchor": {
    "var": "n",
    "domain": "integer",
    "start": 1,
    "end": 300
  },
  "parameters": {},
  "maps": [
    {"id": "x0", "expr": "n"},
    {"id": "x1", "expr": "n^2+n+41"}
  ],
  "predicate": {
    "op": "all_prime",
    "args": ["x0", "x1"]
  },
  "quantifier": {
    "type": "INF_RECURRENCE"
  }
}
```

It demonstrates that

a linear position + a quadratic position

may belong to the same relation object.

Figure 7-07 RSL Syntax Tree

RSL has six principal fields; subfields are contained inside their parent nodes, avoiding fan-out lines through text.

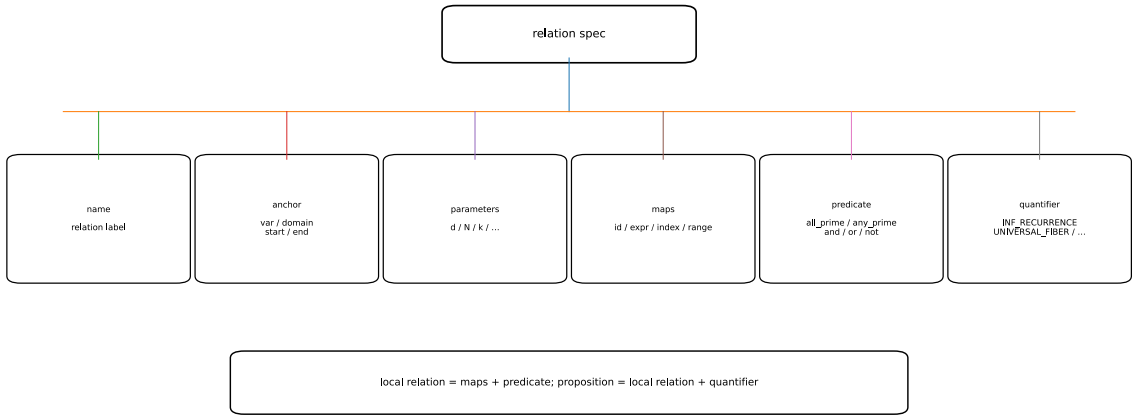


Figure 7-07. RSL Syntax Tree. An RSL object consists of fields such as `name`, `anchor`, `parameters`, `maps` or `map_generator`, `predicate`, and `quantifier`. `anchor` stores the free variable and finite domain, `parameters` stores fixed relation parameters, position maps generate the relation endpoints, `predicate` combines local relations on fixed identities, and `quantifier` lifts the local relation to a proposition. The relation name is only a readable label; execution semantics live in these structural fields.

Chapter 55 The RSL Compilation Pipeline

The execution order of RSL is

$$\boxed{\begin{array}{c}\text{parse JSON}\\ \Downarrow\\ \text{check anchor / parameters}\\ \Downarrow\\ \text{compile maps / generator}\\ \Downarrow\\ \text{estimate maximum read coordinate}\\ \Downarrow\\ \text{construct the fixed spectrum}\\ \Downarrow\\ \text{read identities anchor by anchor}\\ \Downarrow\\ \text{execute predicate}\\ \Downarrow\\ \text{aggregate the current finite readout of the quantifier.}\end{array}} \tag{55.1}$$

The output includes at least

$$\boxed{\text{relation name,}\quad \text{quantifier,}\quad \text{anchor domain,}\quad \text{hits,}\quad \text{max read.}}$$

The itemwise table preserves

$$\boxed{\text{anchor} + \text{positional identities} + \text{local relation symbol.}}$$

55.1 Executable Output of the Formal-Proof Chain

Part VI has already written a classical proposition in the form

$$\boxed{\mathcal{P} \Leftrightarrow \mathcal{Q}_{\Theta,x}[\mathcal{B}_{\Theta}(\boldsymbol{I}(f_1), \dots, \boldsymbol{I}(f_r))].}$$

RSL directly receives the relational normal form on the right-hand side.

A complete execution record may attach

$$\boxed{\Pi_{\text{formal}} = (\text{proposition,maps,predicate,quantifier,certificate,verification}).} \tag{55.2}$$

where:

- `proposition` stores the original proposition;
- `maps` stores the position maps;
- `predicate` stores the local relation predicate on fixed identities;
- `quantifier` stores the global quantifier;
- `certificate` stores the finite or cumulative certificate interface from Part V;
- `verification` stores the currently achieved V_k level.

Thus

$$\boxed{\text{classical proposition} \leftrightarrow \text{relational normal form} \rightarrow \text{RSL spec} \rightarrow \text{finite execution} \rightarrow \text{certificate} \rightarrow} \tag{55.3}$$

55.2 A Relation Name Is Not Execution Semantics

Menu labels such as “Goldbach,” “Twin,” and “Legendre” are human-readable tags only.

Actual execution is determined by

$$\boxed{\text{maps} + \text{predicate} + \text{quantifier}.}$$

Therefore the same classical proposition can be executed either through a preset interface or through a fully equivalent RSL specification.

the formal proofs in the book and custom relations in the software use the same mathematical semantics.

(55.4)

Figure 7-08 RSL Compilation Pipeline

In the execution order of the text: parse → check → compile positions → estimate maximum coordinate → build fixed spectrum → read identities → execute predicate → summarize the current quantifier readout.

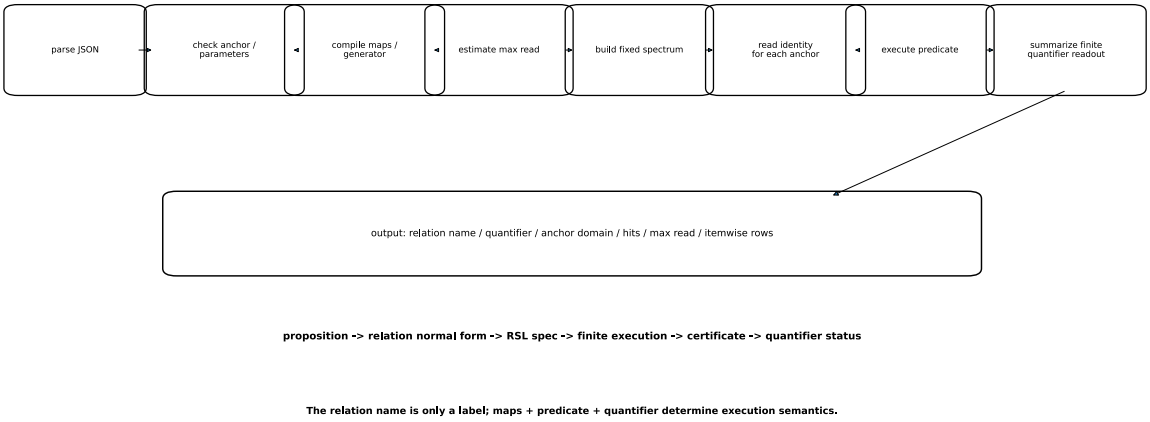


Figure 7-08. RSL Compilation Pipeline. The execution order given in the text is: parse JSON, check `anchor / parameters`, compile `maps / generator`, estimate the maximum read coordinate, construct the fixed spectrum, read identities for each anchor, execute the `predicate`, and finally summarize the current finite readout of the `quantifier`. A complete record may additionally carry proposition, maps, predicate, quantifier, certificate, and verification, forming the outlet chain classical proposition → relation normal form → RSL spec → finite execution → certificate → quantifier status.

Chapter 56 The Open Principle of RSL

If a new relation can be written as

$$\boxed{\text{position maps} + \text{local predicate} + \text{quantifier},}$$

then it already belongs to the unified relational language.

This gives the system an important open discipline:

for a new problem, add a relation definition before adding a new matrix mechanism.

(56.1)

Thus the software is not closed by a list of conjecture names.

Part VII Computational Budget and Execution Backends

Chapter 57 Why the Computational Budget Must Be Estimated First

Different relation tasks have completely different computational shapes:

$$\text{dense full-range sieve} \neq \text{high-coordinate narrow window} \neq \text{BigInt primality} \neq \text{factor sets} \neq \text{ba} \tag{57.1}$$

RenSpectrum therefore does not describe capability by a single “largest integer.”

It first estimates:

- the largest relation coordinate;
- the number of anchors;
- memory required by the dense spectrum;
- task type;
- whether BigInt is involved;
- current device capability;
- recommended backend.

Chapter 58 Memory Model of the Web Dense Spectrum

The current Web engine primarily uses the dense arrays

spf:Int32Array,

with

4 bytes

per integer, and

prime:Uint8Array,

with

1 byte

per integer.

Thus raw arrays use approximately

$$\boxed{M_{\text{raw}}(N) \approx 5(N+1) \text{ bytes.}} \tag{58.1}$$

Typical scales:

Maximum coordinate N	Approximate raw-array memory
10^6	5 MB
5×10^6	25 MB
10^7	50 MB
2×10^7	100 MB
3×10^7	150 MB
10^8	500 MB

Additional memory is consumed by:

- JS objects;
- relation instances;
- temporary arrays;
- the DOM;
- the browser itself;
- garbage-collection peaks.

Therefore

raw-array size is not the actual memory limit.

(58.2)

Chapter 59 Four Task-Risk Levels

Web-side prechecking uses

LIGHT

MODERATE

HEAVY / LOCAL RECOMMENDED

BLOCKED ON WEB.

They mean, respectively:

- suitable for direct execution in the Web page;
- executable, but possibly with noticeable UI load;
- better suited to local CPU / Wasm;
- not executed through the current Web path.

Chapter 60 Web Engineering Thresholds

Hard limit for the dense spectrum:

$$\boxed{N_{\text{dense}}^{\text{hard}} = 3 \times 10^{7}.} \tag{60.1}$$

Hard anchor limit:

$$\boxed{A_{\text{hard}} = 2 \times 10^{6}.} \tag{60.2}$$

Heavy anchor threshold:

$$\boxed{A_{\text{heavy}} = 5 \times 10^{5}.} \tag{60.3}$$

Maximum itemwise DOM rows:

$$\boxed{R_{\text{DOM}} = 5000.} \tag{60.4}$$

DOM truncation affects only display.

Statistics for the complete finite horizon still use every computed result.

Chapter 61 Adaptive Device-Memory Tiers

If the browser exposes

navigator.deviceMemory,

the system adjusts the recommended dense-spectrum scale using coarse tiers:

Browser-visible memory tier	Recommended dense N
≤ 4 GB	6×10^6
≤ 8 GB	1.2×10^7
≤ 16 GB	2×10^7
> 16 GB	2.5×10^7
unknown	10^7

This is an execution-budget tier.

It does not treat the browser's coarse exposed-memory field as an exact measurement of total device memory.

Chapter 62 CPU Threads and Browser Capability Detection

The budget panel displays the number of CPU logical threads exposed by

`navigator.hardwareConcurrency`.

It also detects

`navigator.gpu`

and

`window.crossOriginIsolated`.

These determine, respectively:

- whether WebGPU is available;
- whether the prerequisites for SharedArrayBuffer / Wasm multithreading are present.

Chapter 63 JavaScript Integer Boundary

The largest safe integer of JavaScript `Number` is

$$\boxed{2^{53} - 1 = 9\,007\,199\,254\,740\,991.} \tag{63.1}$$

“Representable as a floating-point number” and “representable exactly as an integer” are different properties.

Any exact arithmetic depending on

$$=, \quad \bmod, \quad |$$

must switch, once this boundary is exceeded, to

$$\boxed{\text{BigInt}}$$

or to a local arbitrary-precision integer library.

Chapter 64 Mersenne and BigInt

The Mersenne readout uses

$$2^p - 1.$$

Its value quickly exceeds the safe-integer range.

The workbench therefore enters the BigInt path directly.

The current hard exponent threshold for the browser Lucas–Lehmer implementation is

$$\boxed{p \leq 5000.} \tag{64.1}$$

This threshold belongs to the Web execution budget.

The mathematical object itself has no such restriction.

Chapter 65 High Coordinates Do Not Imply Large Memory

A relation coordinate may lie near

$$10^{18}.$$

If only the window

$$[10^{18}, 10^{18} + 10^{9}]$$

is read, there is no need to construct a dense spectrum on

$$[0, 10^{18}].$$

RenSpectrum therefore budgets

coordinate magnitude

and

window width

completely separately.

This is the key transition from the Web dense spectrum to a local segmented spectrum.

Chapter 66 Segmented Sieve

The local engine uses

segmented sieve.

A large interval is divided into

$$I_1, I_2, \dots, I_m.$$

Only the sieve state needed for one segment is retained at a time.

Thus memory is reduced from

$$O(N)$$

to the scale

$$O(\text{segment width}).$$

Together with

segment streaming,

prime / factor information can be generated and fed into the relation compiler as it is produced, without retaining the complete fixed spectrum of the entire large range in memory at once.

Chapter 67 Web Worker

Heavy Web computations include:

- sieve;
- factorization;
- relation scan;
- tuple count.

These tasks should not occupy the UI main thread.

The first layer of the high-performance Web edition is therefore

Web Worker.

The interface handles:

- input;
- status;
- visualization.

The Worker handles:

- computation;
- streamed return;
- certificate generation.

Chapter 68 WebAssembly

The number-theoretic core can be compiled as

$$\boxed{\text{Rust / C++} \to \text{WebAssembly.}}$$

The principal advantages of Wasm include:

- performance closer to native CPU;
- more compact data structures;
- SIMD;
- shared algorithms with the Desktop Core;
- easier unified testing.

Chapter 69 Wasm Multithreading

Wasm pthread / SharedArrayBuffer requires proper cross-origin isolation.

Typical deployment headers are:

```
Cross-Origin-Opener-Policy: same-origin
Cross-Origin-Embedder-Policy: require-corp
```

Thus the high-performance Web edition should run under a properly configured HTTPS service.

A single-file HTML edition is more suitable for:

- sharing;
- teaching;
- small- and medium-scale relation readouts.

Chapter 70 The Proper Role of WebGPU

GPU computation is best used as a

hybrid accelerator.

It is particularly suitable for:

Large-Scale Rule Filtering

Residue elimination for a fixed pattern over a large range.

Tuple Counting

When only counts are needed and every prime need not be returned to the UI.

Large-Scale Independent Modular Filters

Large candidate sets subjected simultaneously to the same small-prime conditions.

Chapter 71 The GPU Should Not Carry Every Task

The following tasks are more naturally kept on the CPU:

Outputting Every Prime

After GPU sieving, compaction and GPU-to-CPU transfer are still required.

Arbitrary-Precision BigInt

Consumer GPUs are not automatically efficient general-purpose big-integer engines.

Grimm / Erdős–Woods

Factor sets have variable length, and matching branches are complex.

Arbitrary RSL Expressions

User expressions may involve:

- irregular branching;
- sequential dependence;
- BigInt;
- sparse output.

Therefore

the GPU performs regular parallel work; the CPU handles complex identities and final certification.

(71.1)

Chapter 72 RenSpectrum Desktop

A local edition may use the following architecture:

```
UI
│
├── Desktop Web UI / Tauri
│
└── RenSpectrum Core
    │
    ├── Relation compiler / RSL
    ├── Prime engine
    ├── Segmented sieve
    ├── Big integer / primality
    ├── Factor-set engine
    ├── Balance / seam engine
    └── Scheduler
        ├── CPU backend
        └── GPU backend
```

The specific UI framework is not the essential issue.

What must be shared is

the same relation model / RSL.

Web and Desktop must not maintain two different mathematical semantics.

Chapter 73 Four Execution Backends

Define

Tier W

as Web JS.

Suitable for:

- small- and medium-scale dense readout;
- teaching;
- rapid parameter trials;
- understanding relations.

Tier WW

as Web Worker + Wasm.

Suitable for:

- larger dense / segmented scans;
- multicore CPU;
- streamed relation output.

Tier C

as Local CPU.

With an optimized segmented sieve, the 64-bit integer space can serve as a coordinate interface.

This does not mean a complete scan of

$$[0, 2^{64}].$$

The actual operating modes are:

- high-coordinate narrow windows;
- counting;
- segment streaming;
- relation filtering.

Tier G

as Local CPU + GPU.

Suitable for:

- billion-scale candidate sets;
- fixed patterns;
- large-scale residue filtering;
- massive counting.

Chapter 74 Relation Types and Recommended Backends

Relation	Web	Local CPU	GPU
single-N Goldbach	✓	✓✓	optional
batch Twin / fixed gap	✓	✓✓	✓✓
prime tuple	✓	✓✓	✓✓✓
fixed Green–Tao slice	✓	✓✓	✓✓✓
Legendre / Oppermann	✓	✓✓	optional
large-window Brocard	△	✓✓✓	✓
Andrica / Firoozbakht	✓	✓✓	unnecessary
Mersenne	small p	✓✓✓	specialized implementation
Bunyakovsky / Schinzel	small values	✓✓	depends on expression
Grimm	small scale	✓✓✓	low priority
Erdős–Woods	small scale	✓✓✓	low priority
$B_H/U/\mu/e$	small/medium scale	✓✓✓	B_H scan can be accelerated

Chapter 75 A Reasonable Performance Roadmap

The main bottlenecks of the Web edition can be summarized as

$$\boxed{\text{monolithic dense spectrum} + \text{main-thread compute} + \text{large DOM output.}}$$

The itemwise display cap already controls the third component.

The genuine performance leap comes from

$$\boxed{\text{segment streaming} + \text{Web Worker / Wasm} + \text{native local engine.}}$$

Thus the overall route is

$$\boxed{\text{Web JS} \to \text{Wasm CPU} \to \text{Native CPU} \to \text{Hybrid GPU.}} \tag{75.1}$$

Every tier retains the same fixed-spectrum semantics and the same RSL.

Figure 7-09 Four Execution Backends

The four backends share the same relation model / RSL; only execution capacity and task scale change, not mathematical semantics.

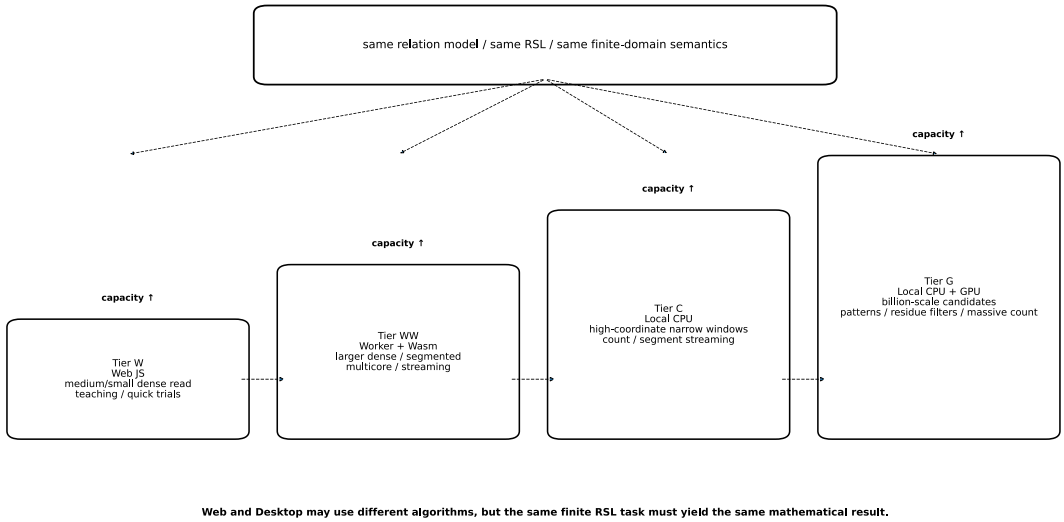


Figure 7-09. Four Execution Backends. The four execution backends are Tier W (Web JS), Tier WW (Web Worker + Wasm), Tier C (Local CPU), and Tier G (Local CPU + GPU). Their appropriate task scales range from medium/small dense reads through larger segmented scans to high-coordinate narrow windows and massive residue/count workloads. Backends may use different algorithms, but Web and Desktop must share the same relation model / RSL rather than maintain different mathematical semantics.

Part VIII Verification Levels

Chapter 76 Why Verification Levels Are Necessary

“The program ran successfully” is not a mathematically precise enough description.

An output may mean only that:

- the JSON parsed;
- one integer was classified correctly;
- one finite relation was read completely;
- one finite certificate was verified;
- one parameter family received a uniform structural certificate;
- one global quantifier was genuinely closed.

These levels must be distinguished.

We therefore define

$$\boxed{V_0, V_1, \dots, V_5.}$$

Chapter 77 V_0: Syntax Is Executable

Requirements:

- parameters are valid;
- RSL parses;
- position expressions compile;
- the domain is enumerable;
- coordinates lie in the representable range of the current backend.

At this stage one obtains only

$$\boxed{\text{the relation definition is executable.}}$$

No mathematical instance has yet been read.

Chapter 78 V_1: Identity Readout Is Independently Checkable

For every read position, the system can return

$$\boxed{A, \quad \mathcal{M}_q, \quad \partial}$$

or the complete

$$\mathrm{PF}(n),$$

and the result can be checked through an independent primality / factorization path.
This is verification at the fixed-spectrum level.

Chapter 79 V_2: Finite Relation Readout Is Independently Checkable

For the current finite horizon, the system completely preserves

$$\boxed{\text{anchor} + \text{positions} + \text{identities} + \text{predicate} + \text{hit/miss}.}$$

and the

$$\boxed{\text{itemwise table}}$$

agrees with the

$$\boxed{\text{aggregate count}}.$$

This is verification at the finite-relation level.

Chapter 80 V_3: Finite Terminal Certificate

One must prove that the current finite object has passed all control layers capable of changing its identity.

For example,

$$z \geq \sqrt{H}$$

completes finite primality certification of the endpoints;

or

$$e_H^{\text{exact}} > 0$$

certifies that a given window contains at least one genuine relation.

At this stage the software does not merely “see a hit”; it can derive an independently verifiable finite certificate.

Chapter 81 V_4: Structural Batch Certificate

A uniform formula covers an entire finite parameter family.

For example,

$$e_H^{\mathrm{struct}} = B_H - \mu_H(Q)(\Delta - 1) > 0$$

may hold across an entire collection of windows.

Or a relation rule may be covered by a uniform state transition for all current parameters.

The verified object has then been lifted from a single instance to

a finite parameter family.

Chapter 82 V_5: Global Quantifier Closure

The highest level must match the target quantifier exactly.

For infinite recurrence, one needs

$$\boxed{C_R(X) \text{ unbounded}}$$

or

$$\boxed{\limsup_{N\to\infty} \mathcal{E}_H(N) = +\infty.}$$

For a universal fiber, one needs

$$\boxed{C(\theta) > 0 \quad \forall \theta.}$$

For window universality, one needs

$$\boxed{W(n) \geq 1 \quad \forall n.}$$

For arbitrary length, one needs

$$\boxed{L(X) \text{ unbounded.}}$$

Therefore

V_5 does not mean ``the computation has gone far enough''; it means that the global quantifier has been closed by a mathematical theorem.

(82.1)

Chapter 83 Basic Discipline of Verification Levels

The verification levels satisfy

$$\boxed{V_0 \prec V_1 \prec V_2 \prec V_3 \prec V_4 \prec V_5.}$$

A higher level must explicitly contain the objects required by the lower levels.

Software output should display

```
the highest verification level currently achieved
```

rather than a vague “verified.”

This keeps

$$\boxed{\text{machine status}}$$

aligned with

$$\boxed{\text{mathematical status.}}$$

83.1 Verification Levels and the Formal-Proof Chain

A standard report for a classical proposition can simultaneously preserve

$$\boxed{\begin{array}{l}\text{original proposition}\\ \text{formally equivalent relational normal form}\\ \text{quantifier type}\\ \text{finite horizon}\\ \text{current certificate}\\ \text{highest verification level.}\end{array}} \tag{83.1}$$

Thus

$$V_0$$

answers whether the relation can be compiled;

$$V_1$$

answers whether the underlying identities can be independently checked;

$$V_2$$

answers whether the current finite relation can be checked item by item;

$$V_3$$

answers whether the current relation has reached finite termination;

$$V_4$$

answers whether a uniform certificate covers the complete finite parameter family;

$$V_5$$

answers whether the global quantifier required by the classical proposition has been closed.

Therefore

the verification level is the execution-side coordinate of the formal-proof chain.

(83.2)

The same proposition may simultaneously possess finite certificates at several levels. The workbench therefore preserves both the semantics of the proposition and the current execution level.

Figure 7-10 Verification Levels V0–V5

Six levels distinguish software success, finite certificates, and global quantifier closure; the strength arrow and note remain fully outside the level nodes.

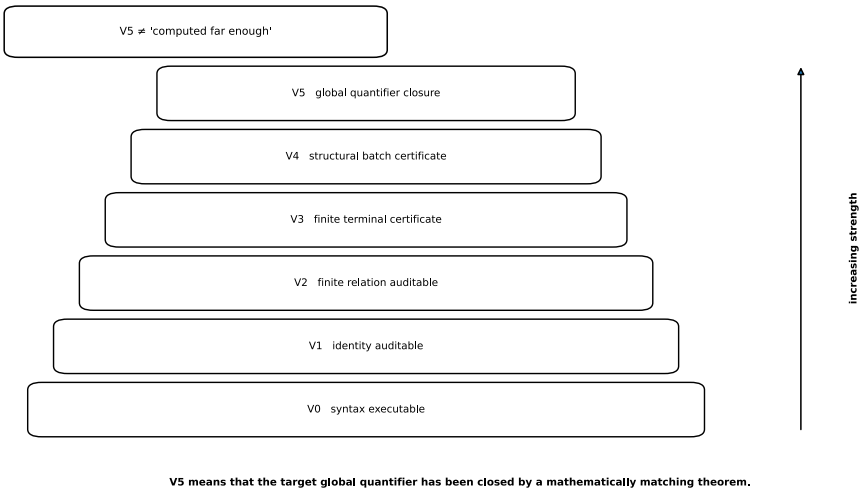


Figure 7-10. Verification Levels V0–V5. The verification levels V_0 through V_5 distinguish, in order: executable syntax, auditable identity readout, auditable finite relation readout, a finite terminal certificate, a structural batch certificate, and global quantifier closure. The highest level V_5 does not mean ‘computed far enough’; it means that the target quantifier—such as infinite recurrence, universal fibers, window universality, or arbitrary length—has been closed by a mathematically matching theorem.

Part IX Machine Checking and Reproducible Experiments

Chapter 84 Determinism First

The foundational relation readouts of RenSpectrum should execute deterministically by default.

The same:

- version;
- RSL;
- parameters;
- backend mathematical semantics;
- finite horizon;

should produce the same relation record and certificate.

If an external algorithm uses randomized primality testing, the following must be recorded:

- algorithm name;
- bases / seed;
- confidence parameter.

At the final certificate layer, the output should whenever possible be converted into a form that can be independently and deterministically verified.

Chapter 85 Minimum Record for Every Run

A complete run record preserves at least

RenSpectrum version
relation name / RSL
parameters
anchor domain
read horizon
control depth z
quantifier type
backend
maximum coordinate
hit count
certificate summary
verification level.

(85.1)

For the balance module one should additionally preserve

$$p, r, s, \Delta, B_H, U, \mu_H, C_{\mathrm{exact}}, C_{\mathrm{struct}}, e_{\mathrm{exact}}, e_{\mathrm{struct}}, T_H, D_H.$$

Chapter 86 Software Version and Relation Version Must Both Be Preserved

The same mathematical relation may be executed in different program versions.

A reproducibility record must therefore preserve both

software version

and

relation specification.

For RSL, the most natural method is to save the complete JSON.

Then even if preset menu names change later, the original relation definition can still be executed again.

Chapter 87 Completeness Checks for Finite Output

Itemwise results may be numerous.

Even if the UI renders only the first

$$5000$$

rows, internal statistics must use the complete finite horizon.

A reproducibility record must therefore distinguish

$$\boxed{\text{computed rows}}$$

from

$$\boxed{\text{rendered rows.}}$$

Display truncation must not alter:

- hits;
- misses;
- B_H;
- T_H;
- D_H;
- certificate status.

Chapter 88 Blocking Examples Are Foundational Regression Tests

The pattern

$$\boxed{H = \{0,2,4\}}$$

must be recognized as completely covered modulo 3.

It is therefore a natural unit test:

$$\boxed{\nu_H(3) = 3.}$$

If the program classifies it as admissible, the local-obstruction module contains an error.

Chapter 89 Relation Identities as Cross-Checkers

The identity from Part III,

$$\sum_{m} |W_K(m) \cap W_K(m+h)| = K\,T_{K,h},$$

can be used as a cross-check between independent implementations.

The left-hand side can be computed from the fixed-sum witness field;

the right-hand side can be computed directly from the fixed-difference relation.

The two paths must agree.

Therefore

exact identities in the theory are naturally also machine-acceptance interfaces for the software.

(89.1)

Chapter 90 Ledger Check for $B_H = T_H + D_H$

Every square horizon must satisfy

$$\boxed{B_H = T_H + D_H.}$$

The system must also check

$$\boxed{D_H \le \mu_H(Q)|U|}$$

and

$$\boxed{T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}.}$$

These are not empirical relations among display fields.

They are mathematical consistency conditions that the program must satisfy window by window.

Chapter 91 The Overlap Region of the BigInt and Number Paths

On small coordinate ranges, the same relation can be implemented simultaneously with

Number

and

BigInt.

Within the safe-integer range

$$n \leq 2^{53} - 1,$$

the two paths should return exactly the same integer results.

This overlap region is an important regression test for the BigInt path.

Chapter 92 Semantic Consistency Between Web and Desktop

Web and Desktop may use different algorithms:

- dense sieve;
- segmented sieve;
- primesieve;
- GMP;
- GPU filtering.

But as long as the relation semantics are the same, the mathematical outputs must agree.

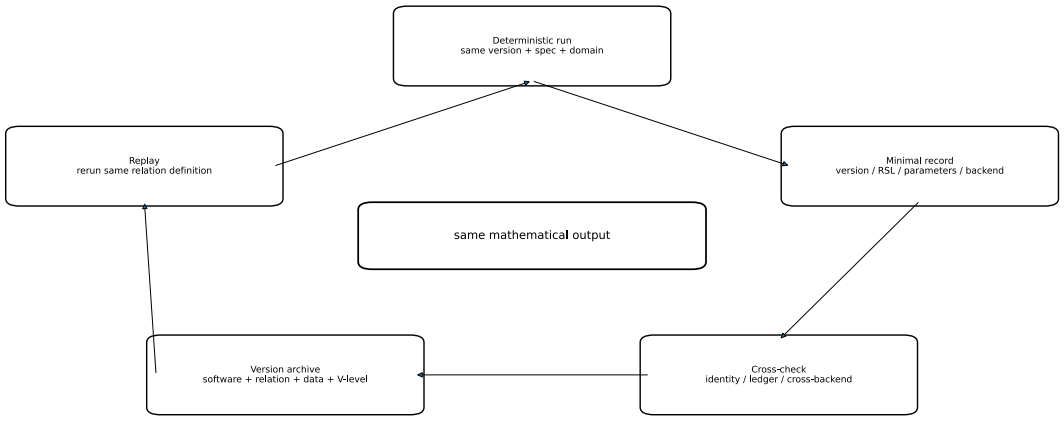
Therefore establish

$$\boxed{\text{same RSL} + \text{same finite domain} \Rightarrow \text{same relation result.}} \tag{92.1}$$

This allows the backend to be optimized freely without changing the theory.

Figure 7-11 Closed Loop for Machine Checks and Reproducible Experiments

Deterministic execution, minimal records, cross-checks, version archiving, and replay form a loop; connectors run around the outside of nodes.



Rendered rows may be truncated; internal finite statistics and certificates may not be truncated by the UI.

Figure 7-11. Closed Loop for Machine Checks and Reproducible Experiments. Reproducible experiments default to deterministic execution: the same version, relation spec, parameters, finite horizon, and mathematical semantics should produce the same relation record and certificate. Each run saves a minimal record and verification level, is cross-checked using obstruction examples, exact relation identities, the $B_H = T_H + D_H$ ledger, the Number/BigInt overlap region, and same-semantics Web/Desktop tasks, and is then archived with both software and relation versions for later replay.

Part X Complete Parameter Dictionary

Chapter 93 Identity-Spectrum Parameters

$$\boldsymbol{I(n)}$$

The complete fixed identity record of the integer n.

$$\boldsymbol{\alpha(n)}$$

$$\alpha(n) = 1$$

means that n is prime.

$$\boldsymbol{P^{-}(n)}$$

The least prime factor.

$$\boldsymbol{\Omega(n)}$$

The total number of prime factors counted with multiplicity.

$$\mathbf{PF}(\boldsymbol{n})$$

The set of distinct prime factors.

$$\boldsymbol{A}$$

The set of atoms.

$$\boldsymbol{\mathcal{M}}$$

The multiplicative occupancy set.

Chapter 94 Relational-Language Parameters

$$\Theta$$

External parameters of the relation.

x / anchor

The free coordinate of a relation instance.

$$f_i$$

Position maps.

$$\mathcal{B}$$

Local relation predicate.

$$Q$$

Outer quantifier.

$$H$$

A fixed-offset pattern.

$$k$$

Pattern length, progression length, or relation-length parameter.

$$d$$

Fixed difference or progression step.

$$N$$

Fixed-sum target or another global target.

$$A, B$$

Coefficients in the affine map

$$Ap + B.$$

Chapter 95 Finite-Horizon and Seam Parameters

$$\boldsymbol{z}$$

Current control depth.

$$\boldsymbol{W_z}$$

The common phase period of the finite control layer:

$$W_z = \prod_{\substack{q \le z \\ q \text{ odd prime}}} q.$$

$$\boldsymbol{p < r < s}$$

Three consecutive odd primes.

$$\boldsymbol{r^2}$$

The left seam of the current square horizon.

$$\boldsymbol{s^2}$$

The next generation seam.

$$\boldsymbol{\Delta}$$

$$\Delta = s - r.$$

Chapter 96 Obstruction Parameters

$$\boldsymbol{\nu_H(q)}$$

The number of distinct residue classes modulo q occupied by the pattern H.

$$\boldsymbol{\nu_H(q)}$$

$$\nu_H(q) = q - \nu_H(q).$$

The number of allowed residue classes.

admissible

Means that

$$\nu_H(q) < q$$

holds for every prime q that needs to be checked.

Chapter 97 Supply–Absorption Parameters

$$\boldsymbol{B_H}$$

Complete old-history relation supply in the current square horizon.

$$\boldsymbol{U(r,s)}$$

The finite cofactor resource pool available for absorption.

$$\boldsymbol{Q}$$

The set of wheel primes used to control the degree of resource reuse.

$$\boldsymbol{M_Q}$$

$$M_Q = \prod_{q\in Q} q.$$

$$\boldsymbol{J_H(c;Q)}$$

The set of pattern roles that can absorb a resource when the resource lies in the wheel unit residue class c.

$$\boldsymbol{\mu_H(Q)}$$

The maximum absorption reuse degree:

$$\mu_H(Q) = \max_c |J_H(c;Q)|.$$

$$\boldsymbol{C_H^{\mathrm{exact}}}$$

$$C_H^{\mathrm{exact}} = \mu_H(Q)|U|.$$

$$\boldsymbol{C_H^{\mathrm{struct}}}$$

$$C_H^{\mathrm{struct}} = \mu_H(Q)(\Delta - 1).$$

$$\boldsymbol{e_H^{\mathrm{exact}}}$$

$$e_H^{\mathrm{exact}} = B_H - C_H^{\mathrm{exact}}.$$

$$\boldsymbol{e_H^{\mathrm{struct}}}$$

$$e_H^{\mathrm{struct}} = B_H - C_H^{\mathrm{struct}}.$$

$$\boldsymbol{T_H}$$

The number of genuine all-atomic relations.

$$\boldsymbol{D_H}$$

The number of genuine deaths:

$$D_H = B_H - T_H.$$

Chapter 98 Quantifier Parameters

$$C_R(X)$$

Cumulative hit count for infinite recurrence.

$$C(\theta)$$

Witness count in a fixed parameter fiber.

$$W(n)$$

Window occupancy count.

$$L(X)$$

Maximum realized relation length within horizon X.

$$P(X)$$

Number of successful parameter fibers.

$$F(X)$$

Cumulative number of failures of a parameterwise universal predicate.

$$K_*$$

Uniform finite block length required for identity-block separation.

Chapter 99 Software-Budget Parameters

max coordinate

Largest integer coordinate that may be read by the current task.

anchors

Number of anchors that must be expanded by the current finite task.

dense bytes

Raw dense-spectrum array estimate:

$$5(N+1) \text{ bytes.}$$

CPU logical threads

The number of logical threads visible to the browser.

browser-visible memory tier

The coarse memory tier exposed by the browser.

WebGPU

Whether the current browser exposes a GPU compute interface.

cross-origin isolation

Whether the prerequisites for SharedArrayBuffer / Wasm threads are satisfied.

DOM row cap

Currently:

$$5000.$$

task risk

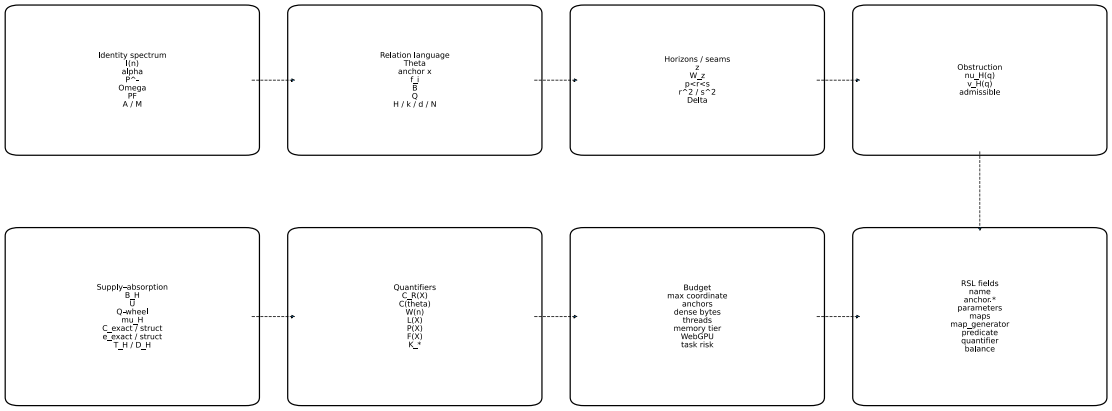
LIGHT | MODERATE | HEAVY | BLOCKED.

Chapter 100 RSL Field Dictionary

Field	Meaning
`name`	relation name
`anchor.var`	anchor variable name
`anchor.domain`	integer / prime / odd / even
`anchor.start`	finite starting point
`anchor.end`	finite endpoint
`parameters`	fixed relation parameters
`maps`	fixed-length position maps
`map_generator`	variable-length position bundle
`id`	map-result identifier
`expr`	integer expression
`predicate`	local relation predicate
`quantifier`	outer quantifier
`balance`	optional obstruction / emission–absorption interface

Figure 7-12 Relationship Graph of the Parameter Dictionary

Parameters are grouped into identity spectrum, relation language, horizons/seams, obstruction, supply–absorption, quantifiers, budgets, and RSL fields; edges indicate execution dependencies, not numerical equivalence.



The dictionary is not a dead list; it describes a dependency network from identity readout to execution and certificates.

Figure 7-12. Relationship Graph of the Parameter Dictionary. The parameter dictionary of Part X falls into eight groups: fixed-identity-spectrum parameters, relation-language parameters, finite-horizon and seam parameters, local-obstruction parameters, supply–absorption parameters, quantifier parameters, software-budget parameters, and RSL fields. They are not isolated lists: identity and relation parameters determine what is read, horizons and obstruction determine finite control, supply–absorption and quantifiers determine certificate structure, budget parameters determine the execution backend, and RSL fields compile these semantics back into executable objects.

Part XI Reproducible Experiment Protocols

Chapter 101 Goal of the Protocols

The goal of a reproducible experiment is not to preserve a screenshot.

What must be preserved is

$$\boxed{\text{whether another person can recover the same mathematical output from the same relation definition.}}$$

The protocol is therefore built around

$$\boxed{\text{definition} + \text{parameters} + \text{backend} + \text{certificate.}}$$

Chapter 102 Protocol A: Fixed-Relation Readout

Preserve:

1. RenSpectrum version;
2. relation name;
3. complete RSL or equivalent built-in spec;
4. parameters;
5. anchor domain;
6. read horizon;
7. control depth;
8. filter;
9. max coordinate;
10. hits / total;
11. verification level.

After re-execution, require

identical itemwise identities

and

identical aggregate counts.

Chapter 103 Protocol B: Obstruction Certificate

Preserve

$$H, \quad q, \quad \nu_H(q), \quad \nu_H(q).$$

If

$$\nu_H(q) = q,$$

the system must output

a complete local-cover certificate.

This can be checked independently by residue enumeration.

Chapter 104 Protocol C: Emission–Absorption Certificate

For each square horizon preserve

$$p, r, s, \Delta.$$

Also preserve

$$B_H, \quad U, \quad Q, \quad M_Q, \quad \mu_H(Q),$$

$$C_H^{\text{exact}}, \quad C_H^{\text{struct}}, \quad e_H^{\text{exact}}, \quad e_H^{\text{struct}}, \quad T_H, \quad D_H.$$

Independent verification must check at least

$$B_H = T_H + D_H,$$

$$D_H \le \mu_H(Q)|U|,$$

$$T_H \ge e_H^{\text{exact}} \ge e_H^{\text{struct}}.$$

Chapter 105 Protocol D: Cumulative Infinitude Interface

Preserve the sequence over consecutive windows

$$e_{H,1}, e_{H,2}, \ldots.$$

Define

$$\mathcal{E}_H(N) = \sum_{j \le N} e_{H,j}.$$

A finite experiment records only the value and trajectory of

$$\mathcal{E}_H(N)$$

at the current N.

If a mathematical theorem further proves

$$\limsup_{N \to \infty} \mathcal{E}_H(N) = +\infty,$$

then the verification level rises to

$$\boxed{V_5}.$$

Chapter 106 Protocol E: BigInt and Large-Integer Certificates

Preserve:

- the integer expression;
- the BigInt decimal or hexadecimal value;
- the primality algorithm;
- certificate parameters;
- modular-arithmetic results;
- the independent verifier result.

For Mersenne numbers one may record

$$p, \quad M_p = 2^p - 1,$$

and the final Lucas–Lehmer state.

Chapter 107 Protocol F: Web / Desktop Cross-Verification

Choose the same finite RSL task.

Execute it using

Web dense / Wasm

and

Desktop segmented.

Compare:

- identities;
- hits;
- misses;
- $B_H/U/\mu/e/T/D$;
- certificates.

Require

complete agreement of mathematical output.

Performance may differ;

semantics may not.

Chapter 108 Protocol G: Version Archiving

For every citable result preserve

software version + relation-definition version + data file + run parameters + verification level.

Where the implementation allows, also attach:

- source commit;
- build id;
- OS / browser;
- CPU;
- memory tier;
- GPU;
- execution time.

These fields support performance reproducibility;

the mathematical core of the output remains the relation spec and its certificate.

Figure 7-13 Seven Reproducibility Protocols

Protocols A–G cover fixed relations, obstruction, supply–absorption, cumulative infinitude, big integers, cross-backend checks, and version archiving.

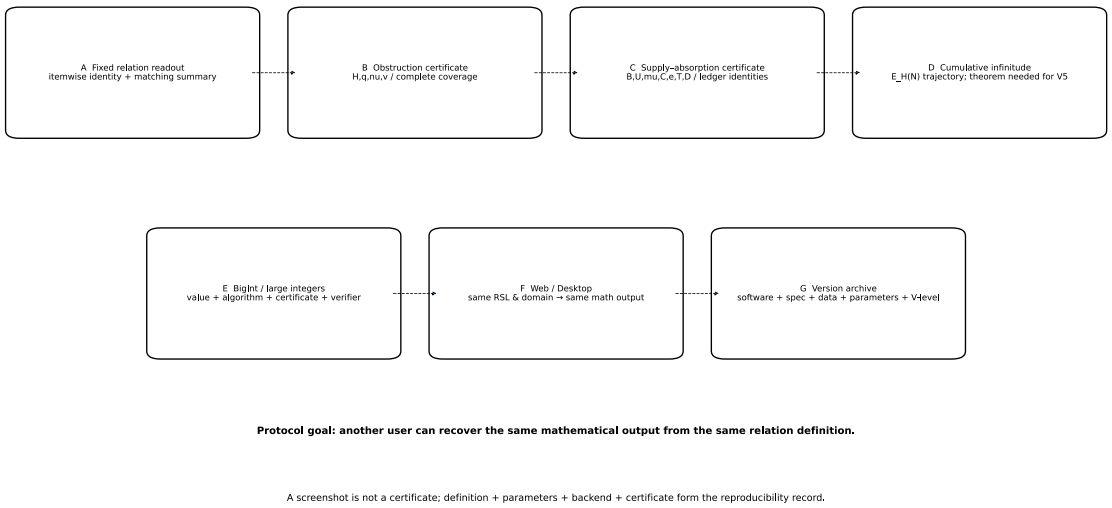


Figure 7-13. Seven Reproducibility Protocols. The seven protocols cover: A, fixed relation readout, requiring itemwise identities and summary counts to agree; B, obstruction certificates, saving $H, q, \nu_H(q), \nu_H(q)$; C, supply–absorption certificates, saving and checking the B, U, Q, μ, C, e, T, D ledger; D, the cumulative infinitude interface, recording the finite trajectory of $\mathcal{E}_H(N)$ while an additional theorem determines any promotion to V_5; E, BigInt and large-integer certificates; F, same-RSL Web/Desktop cross-checks; and G, version archiving of software, relation, data, parameters, and verification level.

Part XII Open Architecture

Chapter 109 Preset Relations Are Not the Boundary of the System

A relation appearing in the menu means only

$$\boxed{\text{it has a convenient preset interface.}}$$

It does not mean

$$\boxed{\text{only those relations belong to the theory.}}$$

The theoretical boundary is determined by

$$\boxed{\boldsymbol{I} + f_i + \mathcal{B} + \mathcal{Q}.}$$

RSL opens this boundary to the user.

Chapter 110 From a “Proof Problem” to a “Compilation Problem”

When facing a new prime-number problem, first ask

$$\boxed{\text{what is its object?}}$$

Then ask

$$\boxed{\text{which identities does it actually require?}}$$

Then ask

$$\boxed{\text{what are its position maps?}}$$

which finite history is already sufficient for certification?

$$\boxed{\text{what is the genuinely difficult quantifier?}}$$

In the first six parts these questions correspond, respectively, to

$$\boxed{\text{object} + \text{relation} + \text{identity} + \text{termination} + \text{quantifier.}}$$

Inside RenSpectrum, they correspond to

$$\boxed{\text{RSL} + \text{control depth} + \text{certificate} + \text{quantifier interface.}}$$

But before execution, one more rigorous mathematical task must be completed:

$$\boxed{\text{original proposition} \Leftrightarrow \text{relational normal form.}}$$

This is the formal proof established in Part VI.

The complete sequence is therefore

$$\boxed{\text{original object} \to \text{relation} \to \text{target identity} \to \text{formal equivalence} \to \text{finite horizon} \to \text{genuine d}} \tag{110.1}$$

Hence

a problem first receives its exact mathematical position; only then are the most suitable proof tool and execution backend chosen.

(110.2)

Chapter 111 What Is Unified Is Not Notation but Information Flow

If a system merely rewrites many problems in the same notation, that is not genuine unification.

What is unified here is

how atoms are read

how relations are compiled

how candidates genuinely die

how finite horizons terminate

how infinite quantifiers close

how certificates are independently checked

and

how classical propositions are translated back without loss.

Therefore

the true advantage of the matrix is not unified notation but unified information flow.

(111.1)

This information flow begins in classical language, enters the relational language through formal proof, passes from RSL into the executable layer, and finally returns through certificates to the mathematical proposition.

RenSpectrum turns this flow of information into an object that can be executed, recorded, and independently checked.

Chapter 112 Why Problems Take the Forms They Do

A mature relation workbench should not answer only:

> *What is the result of computing this instance?*

It should also allow the user to see:

> *Why does this problem naturally have this structure?*

Why is Goldbach a reflection?

Because

$$p + (N - p) = N.$$

Why is Twin a translation?

Because

$$p \mapsto p + 2.$$

Why does a fixed sum have a natural terminal point?

Because fixing the sum imposes a natural upper bound on the endpoints.

Why does a fixed difference have no common terminal point?

Because the entire relation can translate arbitrarily far to the right.

Why do squares become important boundaries?

Because finite primality certification is controlled by the square-root threshold.

Why does local obstruction precede infinite recurrence?

Because one completely covering modular layer already closes every possibility of long-term all-atomic recurrence.

Thus

> **the relational language does not merely execute answers; it also displays why a problem has the shape it has.**

(112.1)

Chapter 113 Openness Does Not Mean Adding Infinitely Many Buttons

Open architecture is easily misunderstood as:

> *Add more menu items.*

True openness means

> **new relations can compile themselves from existing primitives.**

That is the purpose of RSL.

If a user proposes a new relation

$$f_1(n), \dots, f_r(n),$$

the system first asks

> can it be expressed through the existing maps / predicate / quantifier primitives?

If so, the theory does not change.

Only the relation definition is added.

This allows the system to keep growing without continually inflating the matrix itself.

Chapter 114 New Relations Can Also Test the Language of the Theory

Openness is not one-way.

If a new relation cannot be expressed in the current RSL, two possibilities arise.

First:

only a readout primitive is missing.

Then the predicate or map language can be extended.

Second:

the relation genuinely requires an identity field not yet preserved in the fixed spectrum.

Then one should state explicitly which additional arithmetic identity record is required.

A new relation is therefore both a user of the language and a test of the language.

Chapter 115 The Software Does Not Think for the Reader

RenSpectrum can:

- enumerate;
- compile;
- filter;
- count;
- display relation histories;
- produce finite certificates;
- check identities.

But a new mathematical idea still comes from

$$\boxed{\text{a person seeing a relation worth defining.}}$$

The value of an executable system is therefore not to turn mathematics into buttons.

It is to hand mechanical burden to the machine so that people can see more clearly

$$\boxed{\text{objects} + \text{relations} + \text{quantifiers.}}$$

Chapter 116 An Open Workbench Should Preserve Failure

If a relation fails in a given horizon, the system should not merely display

NO.

It should preserve, whenever possible:

- which endpoint failed;
- who the first killer was;
- whether there is a local obstruction;
- whether the current finite horizon merely contains no hit;
- which resource absorbed the relation;
- which quantifier remains undecided.

Therefore

failure is not deletion; failure is also relational knowledge.

(116.1)

This is consistent with the genuine-death viewpoint developed throughout the book.

Chapter 117 From the Fixed Spectrum to Open Knowledge Generation

The book establishes a stable chain:

atoms → relations → histories → horizons → readouts.

Part VI adds

formal proof → translation back to classical propositions.

RenSpectrum further adds

compilation → execution → certificate → reproduction.

The entire system therefore becomes

mathematical structure → formal proof → relational language → executable knowledge → indep (117.1)

Formal proof guarantees that no semantics are lost between the original proposition and its relational normal form; independently checkable certificates guarantee that execution results can return to the mathematical proof chain.

Executable knowledge is therefore not a second system outside the text. It is the operational surface of the theorems in the text.

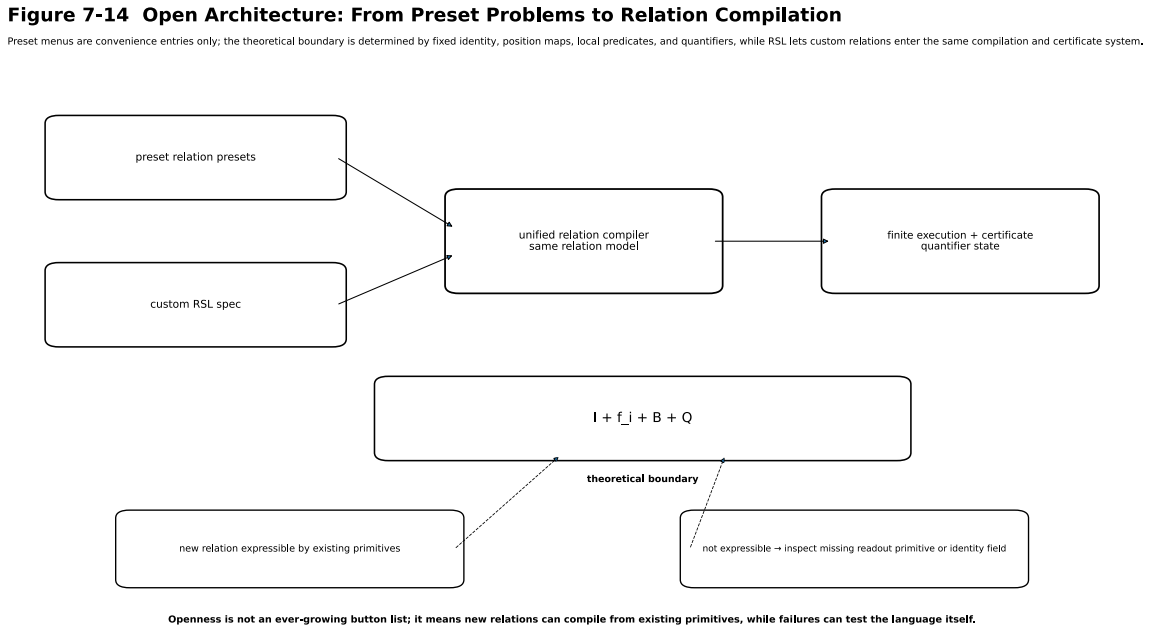


Figure 7-14. Open Architecture: From Preset Problems to Relation Compilation. Preset menus only mean that certain relations have convenient entry points; they do not define the theoretical boundary. The boundary is determined by $\boldsymbol{I} + f_i + \mathcal{B} + \mathcal{Q}$. If a new relation can be expressed by the existing maps, predicate, and quantifier primitives, a custom RSL spec can enter the same compilation, finite-execution, and certificate system as a preset problem. If it cannot be

expressed, one must ask whether only a readout primitive is missing or whether the fixed spectrum lacks a required identity field; new relations therefore test the language in return.

Part XIII Closing the Book

Chapter 118 The Seven Parts Seen Again as One Whole

Part I answers:

how are integer identities and the atomic relation matrix established?

Part II answers:

how are relations permanently preserved inside finite moving horizons?

Part III answers:

how do relations form sum–difference–product geometry and enter unified slices and quantifiers?

Part IV answers:

how does local admissibility form CRT survival trees and enter genuine death and the relation alphabet?

Part V answers:

how do finite relation supply, absorption, and certificates enter global quantifiers exactly?

Part VI answers:

how are classical propositions proved item by item to be rigorously equivalent to the unified relational language?

Part VII answers:

how do formal proofs, relation definitions, certificates, and computation become one open execution system?

The seven parts therefore form

$$\boxed{\begin{array}{c}\text{identity}\\ \Downarrow\\ \text{relations and histories}\\ \Downarrow\\ \text{derived geometry and quantifiers}\\ \Downarrow\\ \text{finite control and terminal identity}\\ \Downarrow\\ \text{certificates and global quantifiers}\\ \Downarrow\\ \text{formal proofs of classical propositions}\\ \Downarrow\\ \text{execution and reproduction.}\end{array}} \tag{118.1}$$

Part VII is therefore not merely a software layer appended after the mathematical text.

It reassembles the mathematical objects of the first six parts into something

$$\boxed{\text{definable} + \text{executable} + \text{verifiable} + \text{reproducible} + \text{open to further extension.}}$$

Chapter 119 Atoms Before Problems, Relations Before Names

The whole book can be compressed into one ordering principle:

atoms precede problems; relations precede names; histories precede statistics; finite termination precedes infinite narration; problems are ultimately different readouts of the matrix.

(119.1)

This statement does not require all problems to become the same.

On the contrary, it requires

every relation to be preserved in its own genuine structure.

A fixed sum remains a fixed sum.

A fixed difference remains a fixed difference.

A window remains a window.

A polynomial remains a polynomial.

A factor set remains a factor set.

Unification does not erase differences.

It only prevents those differences from remaining isolated from one another.

Chapter 120 From “How Can This Be Proved?” to “Why Does the Problem Appear in This Form?”

A mathematical problem often begins with the sentence:

> *How can this be proved?*

The relation matrix asks one more question:

why does this proposition appear in this form?

That question does not weaken proof.

It clarifies the object to be proved.

Goldbach’s fixed sum comes from the sum coordinate of a binary atomic relation.

Twin’s fixed difference comes from the difference coordinate of a binary atomic relation.

Square horizons come from the square-root threshold of finite primality certification.

Natural terminal segments come from the intrinsic endpoint bound imposed by a fixed sum.

The resource pool comes from the actual cofactors left by genuine death.

RSL writes these structures directly into executable relation definitions.

Therefore

when a problem is returned to its true relational position, proof changes from external searching into exact readout of internal structure.

(120.1)

Chapter 121 Classical Methods Are No Longer Isolated from One Another

Sieve methods read local multiplicative occupancy.

The circle method reads fixed-sum frequencies.

The large sieve and average-distribution methods read statistical uniformity in modular structures.

Small-gap methods read finite bands of differences.

Higher-order combinatorial methods read high-dimensional affine structure.

L-functions read the global spectrum.

Primality certificates read the terminal identity of finite objects.

RenSpectrum places all of them in front of the same relational language.

Thus

methods need not replace one another; they may read the same relation matrix from different levels.

(121.1)

Once the object, relation, termination, and quantifier have been made explicit, the choice of method becomes both freer and more precise.

Sieves may continue to sieve;

Fourier analysis may continue to read frequencies;

algebraic number theory may continue to seek common algebraic structures;

combinatorial methods may continue to build matchings and higher-order organization;

computation may continue to expand finite horizons;

certificate methods may continue to compress finite verification.

The methodological discipline of the whole book is therefore retained:

unify the foundation, not the tools; unify the objects, not erase the problems; unify the language, not eliminate creativity.

(121.2)

and

the matrix is not the end of method; it is the beginning of more accurate method selection.

(121.3)

Chapter 122 The Unified Matrix Is Not an Archive of Knowledge

A genuinely open theory cannot merely organize what is already known.

It must permit

new relations to be defined

new slices to be discovered

new certificates to be constructed

new structural maps to be identified.

We therefore retain one of the book's core statements:

the unified matrix is not an archive of completed knowledge; it is a structure capable of continuing to generate new knowledge.

(122.1)

RenSpectrum turns this openness from a written principle into an interface.

Chapter 123 The True Meaning of Openness: A Common Entry Point and Freedom of Method

Openness does not mean:

> *Anything can simply be inserted.*

Openness means that the boundaries are clear enough for something new to know where it should enter.

A new relation can ask:

what are my position maps?

which identity fields do I use?

where is my finite termination?

what is my local obstruction?

which quantifier do I carry?

how can someone else verify my certificate?

As long as these questions can be answered, the relation has a definite position.

That is the most important order provided by an open architecture.

123.1 Different People Can Enter Through Different Doors

What is most worth sharing about the unified relation matrix is that anyone can enter through the mathematics they know best.

A researcher skilled in sieve methods can ask:

how should genuine death layers be counted on a new relation orbit?

A researcher skilled in Fourier analysis can ask:

how does frequency readout of a fixed relation fiber correspond to a finite horizon?

A researcher skilled in algebraic number theory can ask:

can the common algebraic structure of a special orbit generate a relation certificate?

A researcher skilled in combinatorics can ask:

can relation supply and absorption be turned into a more general matching theorem?

A researcher skilled in computation can ask:

which finite-horizon regularities can be fully reproduced and promoted to theorem objects?

A researcher skilled in geometry can ask:

what metrics, topologies, or invariants exist on a high-dimensional prime-gap cell complex?

Therefore

the unified matrix does not take problems away from other researchers; it gives more researchers a common entrance.

(123.1)

123.2 The Mathematical Pleasure of Open Work

The most valuable part of mathematics is not only the final answer.

It also includes:

- finding the genuine problem;
- seeing an object that had not been seen before;
- discovering that two distant methods actually act on the same structure;
- compressing a complicated obstacle into a clear relation;
- constructing a certificate that someone else can independently verify;
- allowing a familiar problem to generate a new mathematical object;
- handing one's method to others so they can continue using it.

What an open system truly shares is therefore

discovering objects together, searching for relations together, constructing methods together, verifying results together, and sharing the pleasure of seeing a problem become progressively clearer.

(123.2)

Chapter 124 Division of Labor Between Machines and People

A fixed spectrum contains a great deal of repetitive work:

- sieving;
- factorization;
- residue scans;
- relation enumeration;
- counting;
- certificate checking.

These tasks are highly suitable for machines.

But machines do not replace

choosing relations worth studying

discovering new structural maps

proposing new ways to close quantifiers

understanding why a formula matters.

Thus

the best form of executable mathematics does not remove people from mathematics; it frees them from repetitive calculation so they can attend more closely to relations.

(124.1)

Machines are especially well suited to preserving

complete relation definitions + itemwise identities + failure reasons + certificates + versions + reproduction paths.

People can focus their attention on

why a relation is worth defining + which structure can truly close the quantifier + whether new transfers exist between diff

What people and machines share, then, is not an answer button but a mathematical work object that is checkable and transferable.

Chapter 125 How a Reader Can Continue Using This Book

The reader need not begin from a named problem.

One may begin from any relation of interest.

First, write down the original objects:

$$x_1, \dots, x_r.$$

Second, write the position maps:

$$F = (f_1, \dots, f_r).$$

Third, specify the target identities or multiplicative depths.

Fourth, write the local relation predicate:

$$\mathcal{B}.$$

Fifth, write the quantifier of the original proposition:

$$\mathcal{Q}.$$

Sixth, prove the bidirectional equivalence between the original proposition and its relational normal form:

$$\boxed{\mathcal{P} \Leftrightarrow \mathcal{Q}[\mathcal{B}(\boldsymbol{I}(f_1), \dots, \boldsymbol{I}(f_r))].} \tag{125.1}$$

Seventh, identify the finite observation region and natural terminal horizon.

Eighth, write the genuine-death mechanism and first killer.

Ninth, if the relation is persistent or batch-type, write the actual relation supply

$$B_H.$$

Tenth, write the absorption resources and reuse degree

$$U, \qquad \mu_H.$$

Eleventh, construct a finite or cumulative certificate and connect it to the target quantifier.

Twelfth, write the same relation in RSL, let RenSpectrum execute the finite part, preserve the certificate, and generate a reproducibility record.

The whole entry path can therefore be compressed to

$$\boxed{\text{object} \to \text{relation} \to \text{identity} \to \text{formal proof} \to \text{horizon} \to \text{death} \to \text{supply–absorption} \to \text{certifi}} \tag{125.2}$$

What the software ultimately seeks to do is not to provide an enormous table of answers, but to

force every problem to state clearly what its genuinely difficult part is.

(125.3)

Chapter 126 From the Web Page to an Open Implementation

RenSpectrum Web is the lightest entry point.

It is suitable for:

- reading;
- teaching;
- small- and medium-scale spectrum readout;
- RSL editing;
- sharing relation definitions.

RenSpectrum Desktop handles:

- segmented spectrum;
- BigInt;
- long-running jobs;
- checkpoints;
- batch RSL;
- local result repositories;
- CPU / GPU scheduling.

Both share

the same mathematical semantics.

A relation can therefore move from a formal proof in the book into RSL, from RSL into the Web interface, and from the Web interface into large-scale local execution without changing its mathematical identity.

The map is

theorem in the book ↔ relational normal form → RSL → Web / Desktop → certificate and repro (126.1)

The Web edition is the lightest public entrance;

Desktop is suited to high coordinates, long windows, BigInt, and batch tasks;

changes in computational scale do not change relation semantics.

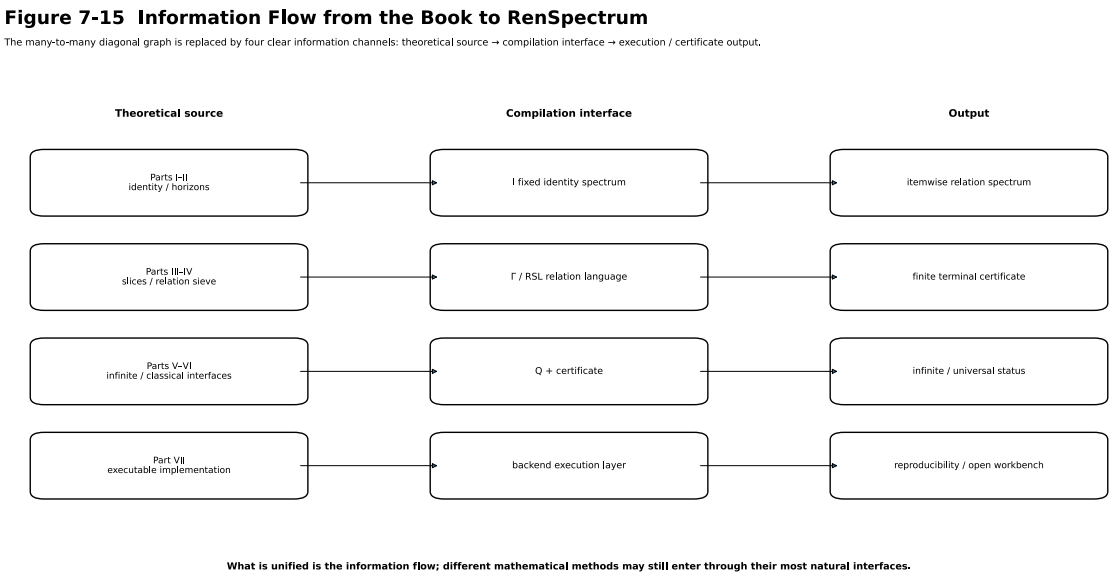


Figure 7-15. Information Flow from the Book to RenSpectrum. Part VII finally compresses the information flow of the book into an execution chain that does not alter mathematical identity: definitions and theorems first enter the unified relation normal form $\boldsymbol{I} + f_i + \mathcal{B} + \mathcal{Q}$, are compiled into RSL, executed as finite tasks by Web or Desktop, and produce identity, relation, terminal, or batch certificates that are stored with version, data, and verification level as reproducibility records. New relations, slices, certificates, and methods can then return through the open interface to the relation-object layer.

Chapter 127 The Final Parameters of the Book Are No Longer Merely Parameters

At this point, parameters are no longer merely a collection of software input boxes.

$$N$$

represents a fixed-sum fiber.

$$d$$

represents a fixed-difference direction.

$$H$$

represents a bundle of fixed-offset relations.

$$z$$

represents the current finite control depth.

$$r^2, s^2$$

represent the two boundaries of a square horizon.

$$B_H$$

represents relation supply.

$$U$$

represents the finite resources available to genuine death.

$$\mu_H$$

represents the reuse capacity of those resources.

$$e_H$$

represents the net relation potential remaining after supply exceeds absorption.

$$\mathcal{E}_H$$

connects finite horizons to infinite quantifiers.
And

$$\mathsf{V}_0, \ldots, \mathsf{V}_5$$

records the mathematical level occupied by the current execution in the formal-proof chain.
A complete parameter set therefore records not only

$$\boxed{\text{what is being computed}}$$

but also

$$\boxed{\text{why it is being computed this way}}$$

and

where the result lies in the proof chain.

Hence

a good program parameter should correspond directly to a clear mathematical object.

(127.1)

The RenSpectrum interface is only the operable surface of these objects.

Chapter 128 Final Conclusion of the Book

If we see only primes, we continually prove propositions about primes.

If we see the fixed identity spectrum, we discover that

the same integer need not be redefined repeatedly in different problems.

If we then see relations, we discover that

the same atoms already carry sum, difference, pattern, window, and higher-order structures simultaneously.

If we then see histories and finite horizons, we discover that

an infinite relation can enter mathematical verification through a sequence of complete finite worlds.

If we separate the quantifiers, we discover that

universality, infinite recurrence, arbitrary length, and parameter infinity are not the same kind of ``infinity.''

If formal proof is added, we discover that

classical propositions can enter the unified relational language without loss and return from the unified language to the orig

If these objects are then written into RSL, we discover that

many new relations do not require rebuilding the matrix; they require an exact statement of their own readout.

The final product of this book is therefore not a closed technique serving only a few classical problems.

It leaves behind

a fixed but nonrigid factual foundation, a rigorous but open relational language, a family of certificate interfaces capable of entering global quantifiers, and a clear path from mathematical propositions to executable verification.

RenSpectrum is the executable entrance to that path.

It lets definitions run, relations expand item by item, failures retain their reasons, certificates be handed to others for independent checking, and new problems acquire their place before proof begins.

More importantly, it turns collaborative work into a concrete mathematical form.

One person can contribute a new position map;

one can prove a new local-obstruction theorem;

one can find a sharper absorption capacity;

one can construct a primality certificate;

one can discover a projection from a higher-order relation to a lower-order one;

another can write all these objects into RSL and ensure that every step leaves an independently checkable record.

They do not need to use the same method, because

they are working on different slices of the same relational world.

(128.1)

The openness left by the book is therefore a common working space that already has explicit objects, language, and interfaces.

We retain two methodological disciplines about openness:

unify the foundation, not the tools; unify the objects, not erase the problems; unify the language, not eliminate creativity.

(128.2)

and

the unified matrix does not take problems away from others; it gives more people a common entrance.

(128.3)

When a new relation appears, it need not first ask:

> *Is there already a button prepared for me?*

It can first ask

what is my relation?

and then write it down.

This is the most human aspect of an open architecture:

the theory does not require later researchers merely to repeat existing problems; it leaves them a place to continue writing.

That place can be continued by different people with different methods.

Mathematics thereby becomes something that can again be shared:

discovering objects together, searching for relations together, constructing methods together, verifying results together, and sharing the pleasure of seeing a problem become progressively clearer.

(128.4)

There need be no struggle between people and tools over who is the master of mathematics.

Machines reliably finish what has already been made precise.

People continue to look for

relations that have not yet been seen.

The book therefore ends not with a stop sign, but with an open door.

atoms precede problems; relations precede names; histories precede statistics; finite termination precedes infinite narration; formal proof guarantees lossless entry of propositions into the unified language; and new relations can continue to arise from here.

How to Use the References in This Book

If you would like to follow the discussions in this book further into the classical literature, standard theorems, and established methods, the references can be read along the following paths.

- If you want to explore the historical background of prime-number research, including the Goldbach–Euler correspondence, Polignac's conjecture, and Landau's four classical problems, begin with [1]–[10], [30]–[32], and [55]–[56].
- If your interest is in the Goldbach problem, the ternary prime problem, Chen-type relations, and the development of the circle method, see [11]–[16] and [51].
- If you want to understand local obstructions, sieve methods, almost-primes, rough numbers, and ideas of finite control, see [17]–[22], [38], and [50].
- If you are interested in Twin primes, fixed gaps, prime tuples, bounded gaps, and arithmetic-progressions of primes, see [23]–[29].
- If you want to explore polynomial orbits, windows, factor sets, and the classical conjectures associated with these structures, see [30]–[36] and [54].
- If you want a broader analytic and spectral perspective on primes—including the ζ-function, L-functions, the Prime Number Theorem (PNT), the Riemann Hypothesis (RH), the Generalized Riemann Hypothesis (GRH), and global spectral interfaces—see [5]–[8] and [37]–[41].
- If you are interested in finite primality certification, machine verification, and the executable ideas behind RenSpectrum, see [42]–[46].

One point is especially important. The framework developed in this book—"Fixed Arithmetic Identity Spectrum – Atomic Relation Matrix – Finite Moving Horizon – Relation Sieve – Quantifier Certificate – RenSpectrum"—is a theoretical construction developed within the book itself.

The purpose of the external references is to help you identify where the classical objects come from, how the historical problems were formed, how standard theorems are stated, and where traditional methods sit within the broader landscape of prime-number research. They do not replace the definitions, theorems, or proof chains developed inside this book.

You may therefore think of the bibliography as connecting two maps.

One map leads into the mathematical tradition that already exists.

The other helps you see where the structures proposed in this book begin, and where they are intended to lead.

Whenever a chapter makes you want to trace its connection to classical mathematics, you can begin with the thematic groups above rather than reading the bibliography sequentially from beginning to end.

References

I. Classical Foundations, the History of Primes, and the Beginnings of Analytic Number Theory

[1] Euclid. *The Thirteen Books of Euclid's Elements*, Vol. II, Book IX, Proposition 20. Translated with introduction and commentary by T. L. Heath. Cambridge University Press, 1908.

[2] Hardy, G. H.; Wright, E. M. *An Introduction to the Theory of Numbers*, 6th ed., revised by D. R. Heath-Brown and J. H. Silverman. Oxford University Press, 2008.

[3] Goldbach, C.; Euler, L. Correspondence of 1742 concerning representations by primes. See: P.-H. Fuss (ed.), *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle*, St. Petersburg, 1843; and the modern historical account in the MacTutor History of Mathematics, "Christian Goldbach."

[4] Euler, L. "Variae observationes circa series infinitas." *Commentarii Academiae Scientiarum Petropolitanae* 9 (1744), 160–188.

[5] Dirichlet, P. G. L. "Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält." *Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften* (1837), 45–81.

[6] Riemann, B. "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse." *Monatsberichte der Berliner Akademie* (1859), 671–680.

[7] Hadamard, J. "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques." *Bulletin de la Société Mathématique de France* 24 (1896), 199–220.

[8] de la Vallée Poussin, C.-J. "Recherches analytiques sur la théorie des nombres premiers. Première partie: La fonction $\zeta(s)$ de Riemann et les nombres premiers en général." *Annales de la Société Scientifique de Bruxelles* 20 (1896), 183–256.

[9] de Polignac, A. "Recherches nouvelles sur les nombres premiers." *Comptes Rendus des Séances de l'Académie des Sciences* 29 (1849), 397–401; erratum, 738–739.

[10] Pintz, J. "Landau's problems on primes." *Journal de Théorie des Nombres de Bordeaux* 21 (2009), no. 2, 357–404.

II. Goldbach, Additive Prime Problems, and the Circle Method

[11] Hardy, G. H.; Littlewood, J. E. "Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes." *Acta Mathematica* 44 (1923), 1–70.

[12] Vinogradov, I. M. "Representation of an odd number as the sum of three primes." *C. R. (Doklady) Acad. Sci. URSS* 15 (1937), 291–294.

[13] Chen, J.-R. "On the representation of a large even integer as the sum of a prime and the product of at most two primes." *Scientia Sinica* 16 (1973), 157–176.

[14] Helfgott, H. A. "The ternary Goldbach conjecture is true." arXiv:1312.7748, 2013.

[15] Helfgott, H. A. *The Ternary Goldbach Problem*. arXiv:1501.05438, 2015.

[16] Nathanson, M. B. *Additive Number Theory: The Classical Bases*. Graduate Texts in Mathematics 164. Springer, 1996.

III. Sieve Methods, Local Obstruction, Almost-Primes, and Distributional Tools

[17] Brun, V. “La série $1/5 + 1/7 + 1/11 + 1/13 + \cdots$, où les dénominateurs sont ‘nombres premiers jumeaux’, est convergente ou finie.” *Bulletin des Sciences Mathématiques* 43 (1919), 100–104, 124–128.

[18] Selberg, A. “On an elementary method in the theory of primes.” *Det Kongelige Norske Videnskabers Selskabs Forhandlinger* 19 (1947), 64–67.

[19] Halberstam, H.; Richert, H.-E. *Sieve Methods*. Academic Press, 1974.

[20] Greaves, G. *Sieves in Number Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 43. Springer, 2001.

[21] Bombieri, E. “On the large sieve.” *Mathematika* 12 (1965), 201–225.

[22] Vinogradov, A. I. “The density hypothesis for Dirichlet *L*-series.” *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 29 (1965), 903–934; correction 30 (1966), 719–720.

IV. Small Gaps, Twin Primes, Prime Tuples, and Arithmetic-Progression Structure

[23] Goldston, D. A.; Pintz, J.; Yıldırım, C. Y. “Primes in tuples I.” *Annals of Mathematics* 170 (2009), no. 2, 819–862.

[24] Zhang, Y. “Bounded gaps between primes.” *Annals of Mathematics* 179 (2014), no. 3, 1121–1174.

[25] Maynard, J. “Small gaps between primes.” *Annals of Mathematics* 181 (2015), no. 1, 383–413.

[26] D. H. J. Polymath. “Variants of the Selberg sieve, and bounded intervals containing many primes.” *Research in the Mathematical Sciences* 1 (2014), Article 12.

[27] Green, B.; Tao, T. “The primes contain arbitrarily long arithmetic progressions.” *Annals of Mathematics* 167 (2008), no. 2, 481–547.

[28] Hardy, G. H.; Littlewood, J. E. “Some problems of ‘Partitio Numerorum’; III.” *Acta Mathematica* 44 (1923), 1–70. (Also an important source for the Hardy–Littlewood prime-tuple heuristic and singular series.)

[29] Schinzel, A.; Sierpiński, W. “Sur certaines hypothèses concernant les nombres premiers.” *Acta Arithmetica* 4 (1958), 185–208; erratum 5 (1959), 259.

V. Classical Prime Relations, Windows, Polynomials, and Factor Identities

[30] Ribenboim, P. *The Little Book of Bigger Primes*, 2nd ed. Springer, 2004.

[31] Ribenboim, P. *The New Book of Prime Number Records*, 3rd ed. Springer, 1996.

[32] Dickson, L. E. *History of the Theory of Numbers*, Vol. I: Divisibility and Primality. Carnegie Institution of Washington, 1919; reprinted by Chelsea.

[33] Grimm, C. A. “A Conjecture on Consecutive Composite Numbers.” *The American Mathematical Monthly* 76 (1969), no. 10, 1126–1128.

[34] Erdős, P. “Quelques problèmes de théorie des nombres.” *Monographies de L’Enseignement Mathématique* 6 (1963), 81–135.

[35] Shorey, T. N.; Tijdeman, R. “Arithmetic Properties of Blocks of Consecutive Integers.” In: J. Sander, J. Steuding, R. Steuding (eds.), *From Arithmetic to Zeta-Functions*. Springer, 2016, 455–471.

[36] Bunyakovsky, V. Ya. “Sur les diviseurs numériques invariables des fonctions rationnelles entières.” *Mémoires de l’Académie Impériale des Sciences de Saint-Pétersbourg*, 1857. (A classical source for prime values of polynomials; see [30][31] for modern discussions.)

VI. Global Spectral Theory, the Prime Number Theorem, and Multiplicative Number Theory

[37] Davenport, H. *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery. Graduate Texts in Mathematics 74. Springer, 2000.

[38] Iwaniec, H.; Kowalski, E. *Analytic Number Theory*. American Mathematical Society Colloquium Publications 53. AMS, 2004.

[39] Montgomery, H. L.; Vaughan, R. C. *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics 97. Cambridge University Press, 2007.

[40] Edwards, H. M. *Riemann's Zeta Function*. Academic Press, 1974; Dover reprint, 2001.

[41] Titchmarsh, E. C. *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown. Oxford University Press, 1986.

VII. Finite Primality Certificates, Algorithms, and Executable Verification

[42] Pocklington, H. C. "The determination of the prime or composite nature of large numbers by Fermat's theorem." *Proceedings of the Cambridge Philosophical Society* 18 (1914), 29–30.

[43] Adleman, L. M.; Pomerance, C.; Rumely, R. S. "On distinguishing prime numbers from composite numbers." *Annals of Mathematics* 117 (1983), no. 1, 173–206.

[44] Atkin, A. O. L.; Morain, F. "Elliptic curves and primality proving." *Mathematics of Computation* 61 (1993), no. 203, 29–68.

[45] Crandall, R.; Pomerance, C. *Prime Numbers: A Computational Perspective*, 2nd ed. Springer, 2005.

[46] Cohen, H. *A Course in Computational Algebraic Number Theory*. Graduate Texts in Mathematics 138. Springer, 1993.

VIII. General Texts, History, and Further Reading

[47] Apostol, T. M. *Introduction to Analytic Number Theory*. Springer, 1976.

[48] Niven, I.; Zuckerman, H. S.; Montgomery, H. L. *An Introduction to the Theory of Numbers*, 5th ed. Wiley, 1991.

[49] Montgomery, H. L.; Vaughan, R. C. "The large sieve." *Mathematika* 20 (1973), 119–134.

[50] Iwaniec, H. "Sieve methods." Proceedings of the International Congress of Mathematicians, Helsinki, 1978, 43–52.

[51] Vaughan, R. C. *The Hardy–Littlewood Method*, 2nd ed. Cambridge University Press, 1997.

[52] Tenenbaum, G. *Introduction to Analytic and Probabilistic Number Theory*, 3rd ed. Graduate Studies in Mathematics 163. AMS, 2015.

[53] Granville, A. "Harald Cramér and the distribution of prime numbers." *Scandinavian Actuarial Journal* 1995, no. 1, 12–28.

[54] Guy, R. K. *Unsolved Problems in Number Theory*, 3rd ed. Springer, 2004.

[55] Narkiewicz, W. *The Development of Prime Number Theory: From Euclid to Hardy and Littlewood*. Springer Monographs in Mathematics. Springer, 2000.

[56] Pintz, J. "Landau's problems on primes." *Journal de Théorie des Nombres de Bordeaux* 21 (2009), 357–404. (A useful general reference for both historical background and modern progress on Landau's four problems.)

Guide to the Use of References

The references above may be used in the main text according to the following principles:

- For the history of primes, the Goldbach–Euler correspondence, Polignac, and Landau's four problems, consult primarily [1]–[10], [30]–[32], and [55]–[56].
- For Goldbach, ternary prime problems, Chen-type relations, and the circle method, consult [11]–[16] and [51].
- For local obstruction, sieve methods, almost-primes, rough numbers, and finite control, consult [17]–[22], [38], and [50].
- For Twin, fixed gaps, prime tuples, bounded gaps, and arithmetic structures in the primes, consult [23]–[29].
- For polynomial orbits, windows, factor sets, and classical conjectures, consult [30]–[36] and [54].
- For ζ, L-functions, the PNT, RH/GRH, and global spectral interfaces, consult [5]–[8] and [37]–[41].
- For finite primality certification, machine verification, and the executable ideas underlying RenSpectrum, consult [42]–[46].

The system developed in this book—"fixed arithmetic identity spectrum → atomic relation matrix → finite moving horizon → relation sieve → quantifier certificate → RenSpectrum"—is the book's own theoretical construction. External references establish the historical origins, standard objects, classical theorems, and traditional methods; they do not replace the internal definitions and proof chains of this book.

Postscript: After Unification, a New Journey Begins

With these seven parts completed, a path from integer identity to classical propositions and then to an executable relational language has been closed into a whole.

But if this book has truly accomplished its aim, this point should not be understood as "the end of prime-number research."

It is better understood as a shift in the center of gravity of research.

In the past, when people spoke of prime-number problems, the first things that came to mind were often a sequence of names: Goldbach, Twin, Polignac, Legendre, Oppermann, Brocard, Bunyakovsky, Schinzel H, Grimm, Erdős–Woods…. Behind every name lies its own history, technical tradition, local breakthrough, and long period of waiting.

That history, centered on named problems, is real and magnificent.

But the order of history need not remain forever the order of theory.

The work carried out across the seven parts of this book is to decompose those names back down to the underlying layers:

identity → atom → relation → horizon → sieve and certificate → quantifier → classical proposition.

Once this chain has been established, the classical problems occupy a different position when we look back at them.

They remain important, but they are no longer the starting point of the theory.

They become readouts of a common matrix.

I. The Seven Parts Are Not Seven Theories, but Seven Layers of One Object

Part I fixes integer identities and establishes atomic generation and full-row birth of relations in the shared odd-number coordinate system.

Part II compresses the infinite matrix into finite moving horizons, giving every finite object a history of genuine death, terminal certification, and permanent preservation.

Part III unfolds relation geometry and places fixed sums, fixed differences, fixed products, affine patterns, finite patterns, arithmetic progressions, windows, and general quantifiers into one relational language.

Part IV enters finite-control geometry and establishes the terminal-saturation relation sieve in the fixed-sum direction.

Part V treats infinite relations with no common terminal horizon, establishing a persistent non-extinction relation sieve and cross-horizon cumulative certificates in the unit fixed-gap direction.

Part VI returns classical propositions and classical methods to the matrix, establishing the complete family of relation slices, norms for formal proof, boundaries of methodological capability, and disciplines governing the transfer of information between higher and lower orders.

Part VII compiles these objects into RenSpectrum, so that the theory can be executed, verified, extended, and recombined.

Thus the seven parts are not a collage of seven ideas.

Together they form one main line:

> **One fixed spectrum gives rise to one relation matrix;**
> **one relation matrix presents different classical problems through different slices and quantifiers.**

II. Goldbach and Twin: The Simultaneous Placement of the Two Most Fundamental Global Relations

Goldbach and Twin occupy a special position in this book.

Not because the other problems are unimportant, but because these two sit at two fundamental extremes of Unified Prime Relations Theory.

Goldbach requires:

> Every fixed-sum fiber must contain a witness.

Twin requires:

> The unit fixed-gap fiber must continue to contain witnesses arbitrarily far out.

The former is targetwise universality; the latter is infinite recurrence.

The former is absolutely accounted for through finite terminal horizons; the latter is continuously certified through moving square horizons.

Accordingly, this book never treats Goldbach and Twin as "two problems, two tricks." Their atomic identities are the same, their relation-birth mechanism is the same, their relation histories are the same, their sum–difference coordinates are the same, and their finite-certification principle is the same; what genuinely differs is the direction of readout and the structure of the quantifier.

For this very reason, their proofs have significance beyond the two individual problems.

To say merely that "two famous conjectures have been solved" would still miss what this book most wants to leave behind.

More important is the following:

> **The two most famous two-atom relations receive, within one and the same matrix,**
> **a unified generative explanation and a unified proof language.**

Once this step is complete, a family of other classical propositions no longer needs isolated underlying objects of its own. Fixed-gap families, weighted reflections, finite patterns, arithmetic-progression chains, window occupancy, polynomial orbits, complete factor sets, and spectral relations can all be located in the same language and can invoke proof and certificate systems matched to their own quantifiers.

In this sense, the achievement of the book is not one isolated "QED."

It is a common grammar behind a family of QEDs.

III. The Real Meaning of "Solving in One Sweep": No Longer Rebuilding a World for Every Conjecture

In mathematics, "conjecture" has never been a word to be treated lightly.

A conjecture means that structure has been seen before it has been proved. The names Goldbach, Polignac, Riemann, Bunyakovsky, Schinzel, and others remain in history precisely because these problems revealed future directions of mathematics in advance.

Thus when this book speaks of moving from "one conjecture after another" toward "a solution in one sweep," it does not mean that the future has no need of conjectures, nor that a single formula can mechanically eliminate all prime-number problems.

What it truly ends is another kind of repetition:

> Whenever a new prime-number problem appears, rebuild an underlying object for it from scratch, explain prime identity all over again, and search for an entirely isolated relational language.

Once Unified Prime Relations Theory is established, a new problem can first be compiled rather than isolated.

It must specify:

1. what the relation orbit is;
2. what identities the endpoints require;
3. where a finite instance receives terminal certification;
4. which information must remain relation-faithful;
5. what the outer quantifier is;
6. how the certificate matches that quantifier;
7. which layer existing methods can act upon.

The name of a conjecture thus changes from "an entire world" into "an address inside the unified matrix."

That is the genuine meaning of "in one sweep."

It does not write all answers in the same sentence; it returns all problems to the same reusable underlying world.

IV. After Proof, the Truly Vast Work Moves to the Back End

In any era, the most visible work is often front-end work: is a proposition true or false? Where is the proof?

Once the underlying objects have been unified, a large amount of new work moves to the back end.

The first task is **formalization**.

The definitions, theorems, boundaries, terminal horizons, seams, quantifiers, and certificates of the unified theory should continue to be compressed into formal objects that can be repeatedly executed by proof assistants, independent programs, and different researchers. A proof should not merely be "readable"; step by step, it should become "replayable."

The second task is **independent verification**.

Goldbach, Twin, and the other new conclusions of this book should not derive their force from the author's identity. A mature result should be detachable from its original exposition and capable of being redefined, rederived, recomputed, and reverified by third parties.

The third task is **certificate compression**.

The existence of a proof does not mean that we have found its best form. Which relation histories are genuinely necessary? Which fields can be compressed safely? What is the shortest finite certificate of a universal proposition? What is the minimal cumulative certificate for an infinite relation? These become new complexity questions.

The fourth task is **expansion of the relation library**.

Two-atom relations are only the beginning. Ternary, quaternary, and general k-ary relations, affine bundles, nonlinear orbits, sparse exponential orbits, mixed additive–multiplicative structures, factor-set matching, and multilayer quantifiers can all continue to enter the unified matrix.

The fifth task is **repositioning traditional methods**.

Sieve methods, the circle method, the large sieve, spectral methods, L-functions, combinatorial methods, probabilistic models, and computational methods do not leave history because of Unified Prime Relations Theory. On the contrary, once the object layer is unified, we can for the first time say more precisely which part of relational information each method preserves and at which layer information is lost.

The sixth task is **opening RenSpectrum**.

The future workbench should not merely display examples. It can become a relation-definition library, a certificate repository, a counterexample search engine, a finite-horizon scanner, a method adapter, an entry point to machine proof, and an open collaboration platform.

At that point, a new prime relation need not begin from an empty page.

It can inherit the entire matrix.

Figure E-02 Back-End Research Roadmap After Proof

After theoretical closure, work shifts to certificates, algorithms, software, data, reproducibility, and discovery; a bus layout avoids diagonal lines through modules.

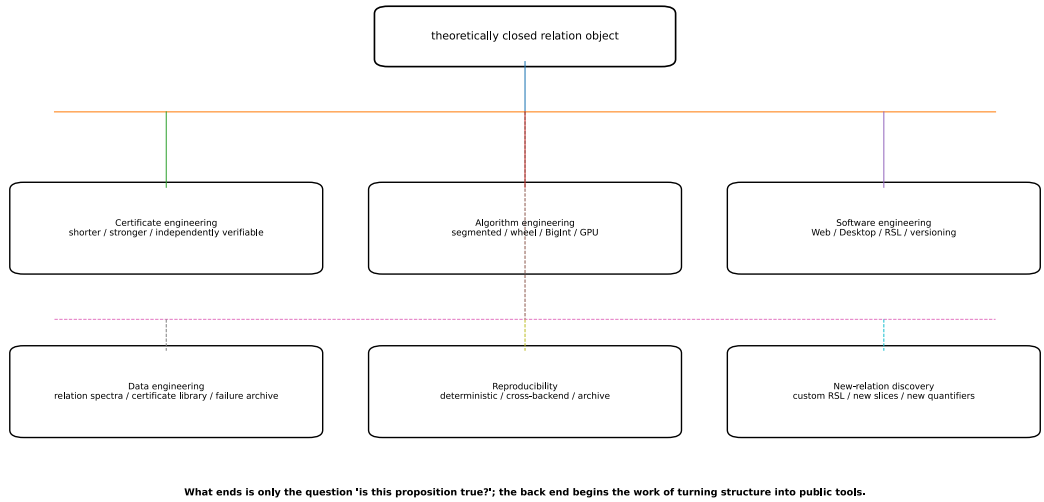


Figure E-02. Six Back-End Routes After Proof. The postscript divides post-proof work into six parallel routes: move definitions, theorems, terminal horizons, seams, quantifiers, and certificates toward replayable formal objects; enable third parties to redefine, rederive, recompute, and reverify results independently; continue compressing finite and cumulative certificates; expand two-atom relations to general k-ary relations, affine bundles, nonlinear/exponential orbits, factor matching, and

multilayer quantifiers; reposition sieve, circle, L-function, combinatorial, spectral, and computational methods by the information they preserve; and open RenSpectrum into a relation library, certificate repository, counterexample search engine, machine-proof entry point, and collaboration platform.

V. Every Era Has Its Own Mission

One of the great missions of ancient Greek mathematics was to transform numerical experience into proof.

The era of Euler, Gauss, Dirichlet, and Riemann advanced prime numbers from discrete observation into series, distribution, congruence, and complex analysis.

The twentieth century created the circle method, sieve methods, the large sieve, exponential sums, L-functions, and increasingly refined statistical and spectral tools.

The twenty-first century has added large-scale computation, formal verification, automated reasoning, and open collaboration to mathematics.

No era advances simply by negating the one before it.

A new mission often emerges when earlier achievements have become rich and powerful enough that they themselves require reorganization.

Prime relations may now stand at such a point.

The earlier task was often:

> *Advance one problem at a time.*

A new task can be:

> Find the common relational world on which these problems depend, so that new problems need not repeatedly rebuild the same foundations.

This does not make mathematics mechanical.

On the contrary, only after the common foundation has been extracted does creativity cease to be spent on repeatedly rebuilding old structures.

Truly new work can then concentrate on genuinely new relations, new quantifiers, new certificates, and new structures.

VI. From a “Catalog of Prime Conjectures” to a “Science of Prime Relations”

If one lists the prime-number problems of history as a catalog, one obtains a sequence of extraordinarily brilliant names.

If one sends them back into the unified relation matrix, another picture emerges:

Goldbach is parameterwise universality of fixed sums;

Twin is infinite recurrence of the unit fixed gap;

Polignac is the full family of fixed even gaps;

Legendre, Oppermann, and Brocard are different window geometries;

Green–Tao is an arbitrary-length arithmetic-progression relation;

Hardy–Littlewood prime tuples are fixed-offset patterns;

Bunyakovsky and Schinzel H are polynomial orbits and polynomial bundles;

Grimm and Erdős–Woods read complete prime-factor identity rather than a single primality indicator;

RH/GRH enters from local relation language into the global spectral interface.

The names do not disappear.

They acquire positions.

When a field moves from “a collection of problem names” toward “a system of objects, relations, and transformations,” it usually means that the field is acquiring a more mature theoretical form.

This is the transformation the book seeks to promote:

from a collection of prime conjectures → a science of prime relations.

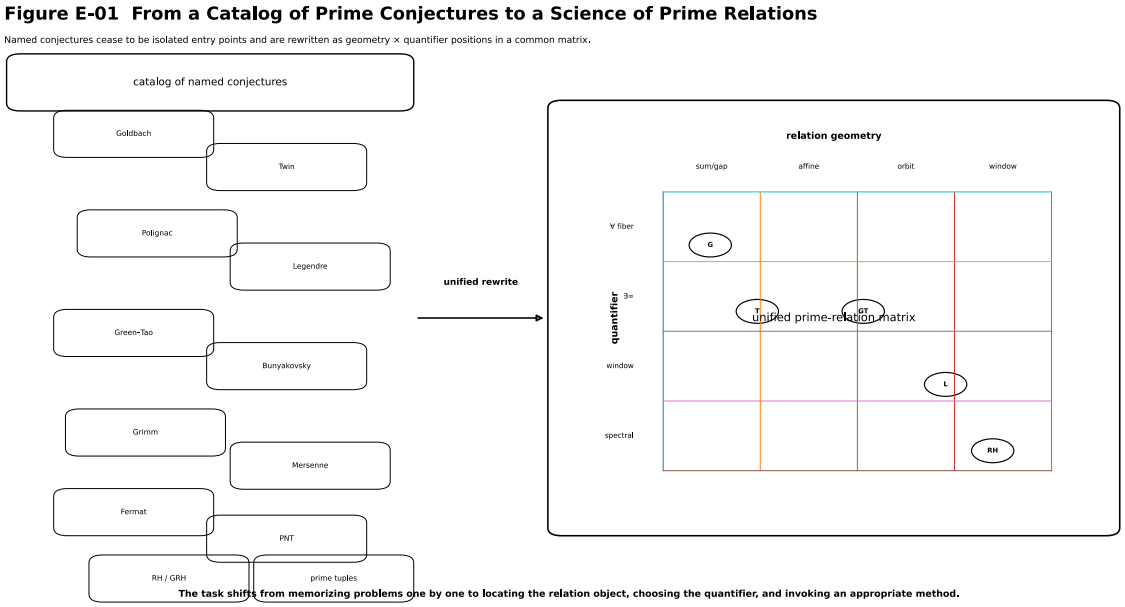


Figure E-01. From a Catalog of Conjectures to a Science of Prime Relations. When the postscript returns named problems such as Goldbach, Twin, Polignac, Legendre, Green–Tao, Bunyakovsky, Schinzel H, Grimm, Erdős–Woods, and RH/GRH to the common matrix, they cease to be merely isolated titles. They acquire positions such as fixed sum, fixed gap, window, affine/pattern, orbit, complete factor identity, and statistical/spectral relation, together with matching outer quantifiers. The names remain, while underlying objects, relational language, and certificate interfaces begin to be shared.

VII. After Unification, the Distance Becomes Clear for the First Time

Unifying prime relations will not strip prime numbers of their mystery.

It merely moves the mystery from “why is this named conjecture so difficult?” to deeper questions:

- What natural geometries of the atomic relation matrix remain unidentified?
- Do new lossless projections, dualities, and conservation laws exist between different relations?

- Can certificate complexity develop a hierarchy theory of its own?
- Which higher-order relations can genuinely transfer witnesses to lower-order relations rather than merely transfer average density?
- Are there adaptive horizons more natural than square horizons or primorial horizons?
- Can relation language itself become a search space for automatic theorem discovery?
- Can an open RenSpectrum allow humans and machines jointly to maintain an ever-expanding arithmetic relation spectrum while preserving complete proof histories?
- Once the relation matrix is extended to higher order, what questions will appear that have not even been named today?

These questions are no longer simple sequels to the old conjectures.

They belong to the new world after unification.

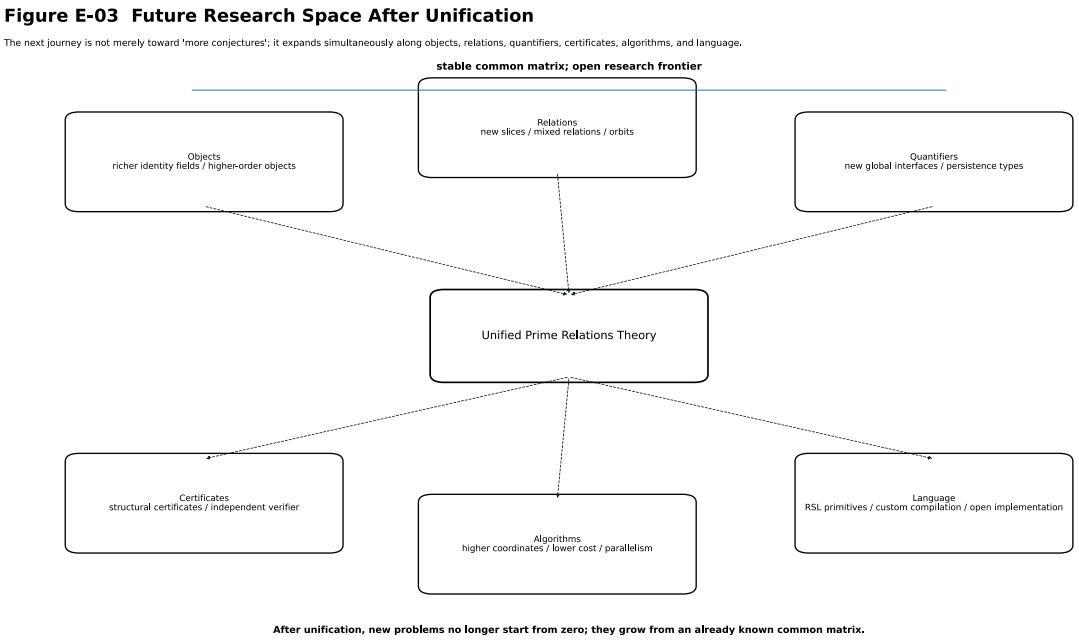


Figure E-03. Future Research Space After Unification. The future questions proposed in the postscript spread across eight directions: identify still-unrecognized natural geometries of the atomic relation matrix; discover new lossless projections, dualities, and conservation laws between relations; develop a hierarchy of certificate complexity; find mechanisms that genuinely transfer higher-order witnesses to lower-order propositions; study adaptive horizons more natural than square or primorial horizons; turn relation language into a search space for automatic theorem discovery; let humans and machines jointly maintain an open relation spectrum with complete histories; and search the higher-order matrix for questions that have not yet been named.

For centuries, Goldbach and Twin stood like two isolated peaks, approached again and again from different directions.

This book attempts something else: not merely to climb the peaks, but also to draw the topography of the mountain range.

Once two peaks are seen to belong to the same continent, their roads are no longer isolated.

And once the map has truly been drawn, the distance does not shrink.

For the first time, the distance becomes visible.

So the last sentence this book wishes to leave behind is not “the problems end here.”

It is:

> **Once prime relations are unified, a new journey has only just begun.**

Every Prime Is Lonely, Yet No Prime Is Truly Isolated

Having finished this book, I do not want it to end with the name Goldbach, Twin, or any other conjecture.

What has drawn me all along is not merely the final answer to a single problem.

For more than two thousand years, people have returned again and again to the primes.

Some have studied how they are distributed.

Some have studied how two primes can add together.

Some have asked whether a fixed gap can occur infinitely often.

Some have pursued sieves, the circle method, analytic functions, probabilistic models, spectra, computation, and ever more refined arithmetic structures.

Different generations have chosen different entrances.

These roads compete with one another, but they also nourish one another.

And if we lift our eyes for a moment from the individual named problems, another picture may come into view:

Goldbach studies a fixed-sum relation.

Twin studies the infinite recurrence of a fixed-gap relation.

Prime arithmetic progressions, window problems, prime templates, exponential orbits, statistical distributions, and spectral structures can likewise be understood anew as—

different readouts of the same arithmetic objects under different relations, different horizons, and different quantifiers.

The purpose of this book, then, is not to compress all mathematics into a single method.

Quite the opposite.

It seeks to preserve the natural place of each method while returning them to a common relational matrix.

A prime number appears lonely.

It cannot be decomposed into a simple product of two smaller non-unit integers. Written alone on the number line, each prime seems to possess its own position, its own distance, its own silence.

People can sometimes seem much the same.

Each person occupies an irreplaceable place and cannot be “factored” out of two other people. Our experiences, memories, loves, and losses do not disappear simply because statistical regularities can be found.

Yet primes have never truly been isolated.

They live together inside the same structure of the natural numbers; through multiplication they rebuild almost the whole arithmetic world; and through sums, differences, windows, orbits, and higher-order relations, they illuminate one another in still other ways.

Studying the relations among primes can sometimes resemble the human search for warmth among other people.

Not because we wish to erase one another’s independence.

But because we want to know that:

independence is not the same as isolation.

A person can remain as irreplaceable as a prime number and still belong to a larger network of relations.

Mathematicians are no different.

Those who work in analytic number theory, sieve theory, combinatorics, spectral theory, probability, or computation—and even those who arrive at completely different judgments about the same question—may all, from different positions, increase the resolution with which mathematics can see.

Mathematics has never flourished merely because everyone walks toward the same door.

It also flourishes because people set out from different directions and, far away, come to see one another.

As this book reaches its end, I also want to leave my thanks to those who have truly been present in the network of relations that makes up my own life.

Thank you to my parents.

Some of the most important forms of support do not always arrive in words. Sometimes they are simply long companionship, understanding, concern, and a quiet support that remains even when I cannot see it.

Thank you to my friends at Matrix.

You were the first readers of my writing and research.

Whenever I made new progress, I would often send it to you first. Sometimes this came after working late into the night, even through an entire night; yet before finally going to rest, I would still send you a screenshot and say: today, I moved a little farther forward.

Most of the time, you were not involved in the mathematical derivations themselves, but you were always there.

That companionship made the long process of writing and research feel less lonely. It also let me know that when an idea was finally written down, or when a problem had finally moved forward, someone would be willing to see it first.

It was this ordinary and quiet companionship that supported me all the way here, and helped me finish this book.

And I am grateful as well to everyone who knows me, and everyone whom I have had the good fortune to know.

Some people have stayed for a long time; others passed through only briefly. Some relationships are clear and deep; others have left only a very small fragment behind.

But every meeting between people changes, in some way, the world each of them inhabits.

If this book has continually asked how “objects appear within relations,” then perhaps my own life is much the same.

No person is made entirely by themselves.

We continue becoming who we are while carrying the language, kindness, misunderstandings, help, arguments, memories, and warmth that others have left with us.

So if this book leaves anything behind, I hope it is not only a collection of proofs concerning Goldbach, Twin, and classical prime propositions.

I hope it also leaves a simpler picture:

every object may preserve its own identity;

every relation deserves to be preserved accurately;

every problem should be understood under the quantifier that properly belongs to it;

and mathematics that appears far apart may, perhaps, have been growing all along in the same soil.

If every person is like a prime number,

then may we preserve what makes each of us irreplaceable,

and still be able to find one another through relation.

Because what we seek has never been only the secret of a number.

Sometimes, through mathematics, we are trying to confirm that:

> **Loneliness can be real, and connection can be equally real.**

Synopsis

This book develops a unified theory of prime relations. It seeks to bring a wide range of classical prime-number problems—long treated as separate questions—back into a common foundational object, a common relational language, and a common structure of finite certification.

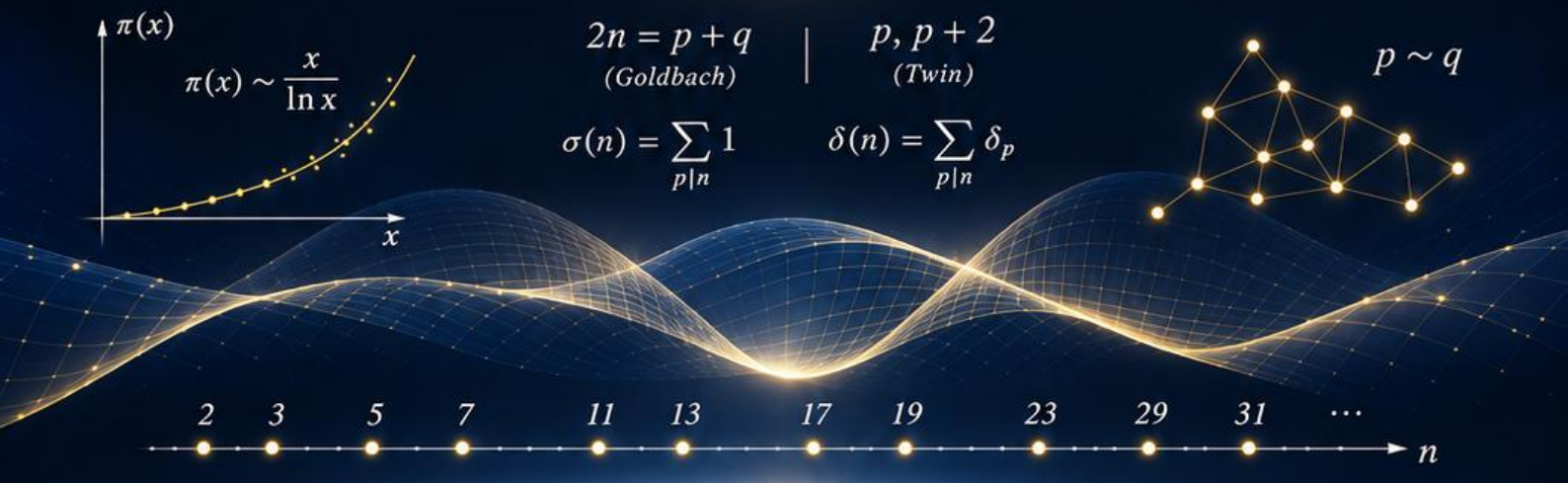
Beginning with the fixed arithmetic identity spectrum, the book constructs an atomic relation matrix in the shared odd-number coordinate system. It then introduces finite moving horizons, relation slices, quantifier structures, and relation sieves, and within this unified framework systematically treats Goldbach, Twin, and a broad family of classical prime propositions.

In this framework, Goldbach is no longer viewed as an isolated fixed-sum problem, and Twin is no longer viewed as an entirely separate fixed-gap problem. Both are returned to the same atomic relation matrix: the former appears as a terminal-saturation relation sieve, while the latter appears as a persistent non-extinction relation sieve. In this way, prime-number problems receive a unified expression through a common matrix and different readings.

Ths seven-part structure unfolds as follows: Part I establishes the fixed arithmetic identity spectrum and the atomic relation matrix; Part II develops finite moving horizons and permanent preservation; Part III develops relation slices and derives the associated geometry and quantifier structures; Parts IV and V provide the core proof chains for Goldbach and Twin; Part VI gathers the formal proofs of classical propositions and the boundaries of methods; Part VII presents the executable RenSpectrum implementation and an open relational language.

This book aims not only to prove major classical propositions, but also to reshape the organization of prime-number research: to unify the underlying object without forcing a single method, and to unify the relational language without erasing the differences between problems.

Once prime relations are unified, a new journey has only just begun.



Yifei Ren

ISBN 978-7-XXXX-XXXX-X
9 787XXX XXXXX